Evaluation of the Feynman Propagator by Means of the Quantum Hamilton–Jacobi Equation

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It is shown that the complex phase of the Feynman propagator is a solution of the quantum Hamilton–Jacobi equation, namely, it is the quantum Hamilton's principal function (or quantum action). Therefore, the Feynman propagator can be computed either by means of the path integration, or by the way of the Hamilton–Jacobi equation. This is analogous to what happens in classical mechanics, where the Hamilton's principal function can be computed either by integrating the Lagrangian along the extremal paths, or as a solution of partial differential equation, namely the classical Hamilton–Jacobi equation. If the path is decomposed in the classical one and quantum fluctuations, the contribution of these quantum fluctuations satisfies a non-linear partial differential equation, whose coefficients depend on the classical action. When the contribution of the quantum fluctuations depend only on the time, it can be computed by means of a simple integration. The final results for the propagators in this case are equal to the Van Vleck–Pauli–Morette expressions, even though the two derivations are quite different.

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1 The Method

As well known, the propagator is the fundamental quantity in Feynman's space-time approach to non-relativistic quantum mechanics [1,2]. It gives the quantum amplitude for a particle to go from a point x_A at time t_A, to x_B at time t_B, as a path integral, namely, a sum of contributions $\varphi[x(t)]$ from each path connecting the two points in the time $t_B - t_A$ (for simplicity we adopt a one-dimensional notation):

$$K(x_B, t_B|x_A, t_A) = \int_{x_A,t_A}^{x_B,t_B} e^{\frac{i}{\hbar}\int_{t_A}^{t_B} L(x(t),v(t),t)\mathrm{d}t} D[x(t)] \quad (1)$$

here $L(x(t), v(t), t)$ is the classical Lagrangian.

The propagator, originally named the kernel by Feynman, is the Green function of the Schrödinger equation [2–6]:

$$\left(\imath\hbar\frac{\partial}{\partial t_B}-H\left(x_B,t_B\right)\right)K\left(x_B,t_B|x_A,t_A\right)=\imath\hbar\delta\left(t_B-t_A\right)\delta\left(x_B-x_A\right) \quad (2)$$

where H is the Hamiltonian operator.

When $x_B \neq x_A$ and $t_B > t_A$, the propagator therefore satisfies the time-dependent Schrödinger equation, and in this way it can be connected to the Quantum Hamilton–Jacobi Equation (QHJE) [7–9]. This latter, fully equivalent to the Schrödinger equation, is the starting point of the Wentzel–Kramers–Brillouin (WKB) approximation, and appears when one tries to find solutions in exponential form

$$\psi(x,t)=Ce^{\frac{\imath}{\hbar}S(x,t)} \quad (3)$$

of the Schrödinger equation for a particle of mass m in a potential $V(x)$:

$$\imath\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi \quad (4)$$

$S(x,t)$ in (3) is a complex function and C is a constant.

By inserting (3) into (4), the time-dependent QHJE results:

$$\frac{\partial S(x,t)}{\partial t}+V(x)+\frac{1}{2m}\left(\frac{\partial S(x,t)}{\partial x}\right)^2-\frac{\imath\hbar}{2m}\frac{\partial^2 S(x,t)}{\partial x^2}=0 \quad (5)$$

When $\hbar = 0$, the last equation reduces to the classical Hamilton–Jacobi equation for Hamilton's principal function S, also named the action [10].

As the propagator actually is a kind of wave function, this argument applies to it too. Therefore, if we write the propagator in the exponential form

$$K\left(x_B,t_B|x_A,t_A\right)=Ce^{\frac{\imath}{\hbar}S\left(x_B,t_B|x_A,t_A\right)} \quad (6)$$

where C is a constant, for $x_B \neq x_A$ and $t_B > t_A$, the quantum action $S(x_B,t_B|x_A,t_A)$ has to satisfy (5) with respect to (x_B,t_B). Thus, if one finds the appropriate solution of (5), by means of (6), it is possible to compute the kernel $K(x_B,t_B|x_A,t_A)$ without do recourse to the path integration. Conversely, from the logarithm of a known propagator, the corresponding solution of (5) can be obtained.

A further step can be done by separating the path $x(t)$ as

$$x(t)=x_{cl}(t)+y(t) \quad (7)$$

where $x_{cl}(t)$ is the classical extremal path, and $y(t_B)=y(t_A)=0$. The quantum action S is therefore split as

$$S\left(x_B,t_B|x_A,t_A\right)=S_{cl}(x_B,t_B|x_A,t_A)+\Delta(x_B,t_B|x_A,t_A)\,, \quad (8)$$

where S_{cl} is the classical action

$$S_{cl}(x_B,t_B|x_A,t_A)=\int_{t_A}^{t_B}L\left(x_{cl}(t),v_{cl}(t),t\right)\mathrm{d}t\,, \quad (9)$$

and the additive term $\Delta(x_B,t_B|x_A,t_A)$ gives the contributions from the expansion of $S(x_B,t_B|x_A,t_A)$ in terms of $y(t)$ and $\frac{dy}{dt}$ (except for the linear one, which vanishes due to the equation of motion).

As a function of (x_B,t_B), the classical action (9) satisfies the classical Hamilton–Jacobi equation (Eq. (5) with $\hbar = 0$). Therefore, by inserting (8) with $x \equiv x_B$ and $t \equiv t_B$ considered as variables and fixed x_A, t_A, into (5), one gets an equation for the function $\Delta(x,t|x_A,t_A)$:

$$\frac{\partial\Delta}{\partial t}+\frac{1}{2m}\left(2\frac{\partial S_{cl}}{\partial x}\frac{\partial\Delta}{\partial x}+\left(\frac{\partial\Delta}{\partial x}\right)^2\right)-\frac{\imath\hbar}{2m}\left(\frac{\partial^2 S_{cl}}{\partial x^2}+\frac{\partial^2\Delta}{\partial x^2}\right)=0 \quad (10)$$

The known quantities in this equation depend on the classical action.

In the following, the dependence of the various functions on the parameters (x_A,t_A) will be sometimes understood.

When Δ depends only on t, which happens for instance when the Lagrangian is quadratic, Eq. (10) reduces to

$$\frac{\partial\Delta}{\partial t}-\frac{\imath\hbar}{2m}\frac{\partial^2 S_{cl}}{\partial x^2}=0 \quad (11)$$

So that the function $\Delta(t)$ in this case is obtained as

$$\Delta(t)=\frac{\imath\hbar}{2m}\int_{t_A}^{t}\frac{\partial^2 S_{cl}}{\partial x^2}\mathrm{d}t\,. \quad (12)$$

Putting this into (8), the propagator is given by (6), apart from the multiplicative constant C which, following Feynman [2], can be computed by remembering that the kernel has to satisfy the equation

$$\psi(x_B,t_B)=\int_{-\infty}^{\infty}K\left(x_B,t_B|x_A,t_A\right)\psi(x_A,t_A)\mathrm{d}x_A \quad (13)$$

The expansion of this equation in the quantities $\epsilon = t_B - t_A$ and $\eta = x_B - x_A$ gives

$$\psi(x,t)+\epsilon\frac{\partial\psi}{\partial t}=\int_{-\infty}^{\infty}K(x+\eta,t+\epsilon|x,t)\left(\psi(x,t)+\eta\frac{\partial\psi}{\partial x}+\eta^2\frac{\partial^2\psi}{\partial x^2}\right)\mathrm{d}\eta\,, \tag{14}$$

where the notation has been simplified by writing (x,t) instead of (x_A,t_A). By doing the integration and comparing the leading terms in ϵ on both sides, the constant C can be computed.

2 Applications

2.1 The free particle

The classical action is:

$$S_{cl}(x,t|x_A,t_A)=\frac{1}{2}\frac{m(x-x_A)^2}{(t-t_A)}\,. \tag{15}$$

From (12) one gets:

$$\Delta(t)=\frac{\imath\hbar}{2}\log\left[t-t_A\right]\,. \tag{16}$$

According to (8) the quantum action, namely, the phase of the kernel, therefore is:

$$S(x,t|x_A,t_A)=\frac{1}{2}m\frac{(x-x_A)^2}{(t-t_A)}+\imath\hbar\log\left[\sqrt{t-t_A}\right]\,, \tag{17}$$

and from (6), the kernel itself

$$K(x,t|x_A,t_A)=C\exp\left[\frac{\imath}{\hbar}\left(S_{cl}(x,t)+\Delta(t)\right)\right]=C\frac{\exp\left[\frac{\imath}{\hbar}\frac{1}{2}m\frac{(x-x_A)^2}{(t-t_A)}\right]}{\sqrt{t-t_A}}\,. \tag{18}$$

At the leading order in ϵ, Eq. (14) gives

$$\psi(x,t)=C\frac{\int_{-\infty}^{\infty}e^{\frac{\imath m\eta^2}{2\hbar\epsilon}}\mathrm{d}\eta}{\sqrt{\epsilon}}\psi(x,t) \tag{19}$$

so that

$$C=\sqrt{\frac{m}{2\pi\imath\hbar}} \tag{20}$$

when inserted into the last equality in (18), reproduces the correct result [2].

2.2 The harmonic oscillator

The classical action is:

$$S_{cl}(x,t)=\frac{m\omega}{2\sin\left[\omega\left(t-t_A\right)\right]}\left[(x-x_A)^2\cos\left[\omega(t-t_A)\right]-2xx_A\right]\,. \tag{21}$$

From (12)

$$\Delta(t)=\frac{\imath\hbar}{2}\log\left[\sin\left[\omega(t-t_A)\right]\right]\,. \tag{22}$$

The quantum action therefore is

$$S(x,t)=\frac{1}{2}m\omega x^2\cot\left[\omega(t-t_A)\right]+\frac{1}{2}\imath\hbar\log\left[\sin\left[\omega(t-t_A)\right]\right] \tag{23}$$

and the kernel is evaluated as

$$K(x,t|x_A,t_A)=C\frac{\exp\left[\frac{\imath}{\hbar}\frac{m\omega}{2\sin[\omega(t-t_A)]}\left[(x-x_A)^2\cos\left[\omega(t-t_A)\right]-2xx_A\right]\right]}{\sqrt{\sin\left[\omega(t-t_A)\right]}}\,. \tag{24}$$

The constant C in this case is

$$C=\sqrt{\frac{m\omega}{2\pi\imath\hbar}}\,. \tag{25}$$

With this value of the multiplicative constant, Eq. (24) gives the well-known propagator for the harmonic oscillator. The procedure above is simpler than the original one by Feynman [2], which integrates on the coefficients of the Fourier expansion for the kernel, and also with respect to that reported in many books [3–6], which exploits the limit of a difference equation.

2.3 The driven harmonic oscillator

For the harmonic oscillator driven by a time-dependent force, the classical action is given by Eq. (3.66) of Ref. [2], and it is the sum of the corresponding one for the free harmonic oscillator, plus terms which are linear in the coordinates x_A and x_B. Therefore, the function $\Delta(t)$ and the constant C are the same as for the undriven case, given by (22) and (25), respectively. In this case too, the propagator from our method is the same as computed by means of the path integration [4].

2.4 Quadratic Lagrangian with time-depending coefficients

The method presented in Section 1 can be applied to the generalization of the previous case, namely, when all the coefficients of the quadratic Lagrangian depend on the time

$$L = \frac{1}{2}\left[a(t)v^2 - b(t)x^2\right] + c(t)x \tag{26}$$

In the following, we will present the case of a damped oscillator [4], with the coefficients

$$a(t) = me^{\gamma t}\ ,\ b(t) = m\omega^2 e^{\gamma t}\ ,\ c(t) = 0\ . \tag{27}$$

The classical Lagrangian is

$$S_{cl}(x,t) = \frac{1}{2}\frac{m}{\sin\left[\Omega\,(t-t_A)\right]}\left\{\frac{1}{2}\left(e^{\gamma t_A}x_A^2 - e^{\gamma t}x^2\right)\gamma + \Omega\left(\left(e^{\gamma t_A}x_A^2 + e^{\gamma t}x^2\right)\cos\left[\Omega(t-t_A)\right] - 2e^{\frac{1}{2}\gamma(t+t_A)}xx_A\right)\right\} \tag{28}$$

where

$$\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}\ . \tag{29}$$

The additive imaginary part $\Delta(t)$ to the quantum action is

$$\Delta(t) = -\frac{1}{4}\imath\hbar\gamma(t-t_A) + \frac{1}{2}\imath\hbar\log\left[\sin\left[\Omega(t_B-t_A)\right]\right]\ . \tag{30}$$

Finally, by computing the multiplicative constant C by the method previously exposed, the propagator for this case results

$$K(x,t|x_A,t_A) = e^{\gamma\frac{t_A}{2}}\sqrt{\frac{m\Omega}{2\pi\imath\hbar}}\frac{e^{\frac{\gamma(t-t_A)}{4}}}{\sqrt{\sin\left[\Omega\,(t-t_A)\right]}}e^{\frac{\imath}{\hbar}\left[\frac{1}{2}m\Omega\frac{1}{\sin[\Omega(t-t_A)]}\left(\left(e^{\gamma t_A}x_A^2+e^{\gamma t}x^2\right)\cos[\Omega(t-t_A)]-2e^{\frac{1}{2}\gamma(t+t_A)}xx_A\right)+\frac{1}{4}m\gamma\left(e^{\gamma t_A}x_A^2-e^{\gamma t}x^2\right)\right]}, \tag{31}$$

which is the correct expression [4].

3 Concluding Remarks

In this paper, we have analyzed the link between the Feynman propagator and the quantum Hamilton–Jacobi equation. When the propagator is written in exponential form, its complex phase is a solution of the QHJE, with respect to the coordinates and time of the final point, and contains the coordinates and time of the initial point as parameters. By analogy with the classical case, we name it the quantum Hamilton principal function or quantum action. Thus, the propagator can be computed either by means of Feynman's path integration, or by means of that equation. This is analogous to what happens in classical mechanics, where Hamilton's principal function can be computed either by integrating the Lagrangian with respect to the time along each extremal path, in a region covered by a family of non-intersecting extremal paths [11, 12], or as solution of a partial differential equation, namely, the classical Hamilton–Jacobi equation.

In classical mechanics the action is a fundamental dynamical quantity. The same happens in the quantum case, being the corresponding quantity, namely, the quantum action, the phase of the wave function, or the phase of the propagator, as the case. These quantum functions generate the corresponding classical ones in the limit when Planck's constant tends to zero, $h \to 0$.

If the path is decomposed in the classical one and the quantum fluctuations, the contribution of these latter satisfies the non-linear partial differential equation (10), whose coefficients depend on the classical action. While this latter is a real function, the quantum one is a complex quantity. When Planck's constant h goes to zero, the imaginary part of the quantum action vanishes and this function reduces to the classical corresponding one. As the Hamilton–Jacobi formulation of the classical mechanics is fully equivalent to those respectively based on the Lagrangian or the Hamiltonian functions, this shows how in this approach the classical mechanics emerges from the quantum one. In fact, these three formulations of the classical mechanics are all based on the same fundamental principle, namely, Hamilton's principle of least action.

When the contribution of the quantum fluctuations depends only on the time, as happens for the quadratic Lagrangian, it is computed by means of a simple integration. It this case, the final result for the propagator is the same of the so called Van Vleck–Pauli–Morette expression [13–15], even if the procedure is quite different.

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