

Canonical Structure of A and B Maps

Sudha¹, B. N. Karthik^{2,3}, A. R. Usha Devi^{2,3} & A. K. Rajagopal³

¹ Department of Physics, Kuvempu University, Shankaraghatta, Shimoga, India. E-mail: tthdrs@gmail.com

² Department of Physics, Bangalore University, Bangalore, India.

³ Inspire Institute Inc., Alexandria, Virginia, USA.

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In their seminal 1961 paper, Sudarshan, Mathews and Rau investigated properties of the dynamical A and B maps acting on n dimensional quantum systems. The nature of dynamical maps in open quantum system evolutions has attracted great deal of attention in the later years. However, the novel paper on the A and B dynamical maps has not received its due attention. In this tutorial article, we review the properties of A and B forms associated with the dynamics of finite dimensional quantum systems. In particular, we investigate a canonical structure associated with the A form and establish its equivalence with the associated B form. We show that the canonical structure of the A form captures the completely positive (not completely positive) nature of the dynamics in a succinct manner. This feature is illustrated through physical examples of qubit channels. *Quanta* 2021; 10: 34–41.

1 Introduction

The conceptual formulation of dynamical A and B forms was pioneered by Sudarshan and coworkers 60 years ago [1,2]. The A and B matrices play an important role to identify if the open system dynamics of finite dimensional

quantum systems is completely positive or not [3–6]. In this article, we study the dynamical A and B forms in detail. In particular, we investigate the canonical structure of the A form and its properties. We show that there is a one-to-one connection between the canonical A form and the B form. We also construct the canonical A form associated with some important physical examples of qubit channels [3].

In Section 2, we introduce the A and B forms and discuss their properties [1, 2]. We present the canonical structure of the A form and establish its equivalence with the B form in Section 3. The canonical structure of the A form is explicitly constructed for several qubit channels in Section 4. Concluding remarks are given in Section 5.


2 Properties of A and B maps

Consider a n dimensional Hilbert Space \mathcal{H}_n . State of a quantum system is described by a density matrix $\rho \in \mathcal{H}_n$, defining properties of which are given by

1. Hermiticity: $\rho^\dagger = \rho$.
2. Unit trace: $\text{Tr} \rho = 1$.
3. Positivity: $\langle \psi | \rho | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$.

For a qubit (two-level quantum system) we have

$$\rho = \frac{1}{2} (I + \vec{\sigma} \cdot \vec{p}) \quad (1)$$

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where I denotes 2×2 identity matrix, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ by $\vec{p} = (p_1, p_2, p_3)$ satisfy the condition are Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$|\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2} \leq 1$. Thus, the state space of a qubit corresponds to a unit ball in \mathbb{R}^3 . The vector \vec{p} is called the Bloch vector.

(2) Now we consider a linear map A trans- and the real qubit state parameters given forming density matrices in \mathcal{H}_n :

$$A : \quad \rho_i \rightarrow \rho_f = A(\rho_i),$$

$$(\rho_f)_{r's'} = (A\rho_i)_{r's'} = \sum_{r,s=1}^n A_{r's';rs} (\rho_i)_{rs}, \quad r', s' = 1, 2, \dots, n. \quad (3)$$

- The $n^2 \times n^2$ matrix A is called a trace-preserving positive map if, for every input density matrix, the output $\rho_f = A(\rho_i)$ is also a legitimate density matrix [1, 2].

Writing the qubit density matrix (1) explicitly in the standard basis $|0\rangle = (1, 0)^T$, $|1\rangle = (0, 1)^T$ as

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & 1 - p_3 \end{pmatrix} \quad (4)$$

one may identify the action $A : \rho_i \rightarrow \rho_f = A(\rho_i)$ (see (3)) as follows:

$$\begin{pmatrix} (\rho_f)_{00} \\ (\rho_f)_{01} \\ (\rho_f)_{10} \\ (\rho_f)_{11} \end{pmatrix} = \begin{pmatrix} A_{00;00} & A_{00;01} & A_{00;10} & A_{00;11} \\ A_{01;00} & A_{01;01} & A_{01;10} & A_{01;11} \\ A_{10;00} & A_{10;01} & A_{10;10} & A_{10;11} \\ A_{11;00} & A_{11;01} & A_{11;10} & A_{11;11} \end{pmatrix} \begin{pmatrix} (\rho_i)_{00} \\ (\rho_i)_{01} \\ (\rho_i)_{10} \\ (\rho_i)_{11} \end{pmatrix}$$

or

$$\begin{pmatrix} 1 + (p_f)_3 \\ (p_f)_1 - i(p_f)_2 \\ (p_f)_1 + i(p_f)_2 \\ 1 - (p_f)_3 \end{pmatrix} = \begin{pmatrix} A_{00;00} & A_{00;01} & A_{00;10} & A_{00;11} \\ A_{01;00} & A_{01;01} & A_{01;10} & A_{01;11} \\ A_{10;00} & A_{10;01} & A_{10;10} & A_{10;11} \\ A_{11;00} & A_{11;01} & A_{11;10} & A_{11;11} \end{pmatrix} \begin{pmatrix} 1 + (p_i)_3 \\ (p_i)_1 - i(p_i)_2 \\ (p_i)_1 + i(p_i)_2 \\ 1 - (p_i)_3 \end{pmatrix}. \quad (5)$$

- Unitary dynamics $\rho_f = U \rho_i U^\dagger$ defines a trace-preserving positive map

$$A_U : \quad \rho_i \rightarrow \rho_f = A_U(\rho_i),$$

$$(\rho_f)_{r's'} = \sum_{r,s=1}^n (U \otimes U^*)_{r's';rs} (\rho_i)_{rs}, \quad r', s' = 1, 2, \dots, n. \quad (6)$$

- Matrix transposition given by

$$A_T : \quad \rho_i \rightarrow \rho_f = A_T(\rho_i) = \rho_i^T \quad (7)$$

is an example of trace-preserving positive map.

Given a positive map A on \mathcal{H}_n one may extend it to the map $A \otimes \mathbb{I}_d$ acting on the tensor product space $\mathcal{H}_n \otimes \mathcal{H}_d$ of a composite $n \times d$ system, where \mathbb{I}_d denotes identity map on \mathcal{H}_d . Then the positive map A acts only on the n -dimensional subsystem of the composite state. If the $A \otimes \mathbb{I}_d$ transforms a composite density matrix into a physical state for any d , then the map A is said to be *completely positive* (CP). Otherwise, it is called *not-completely positive* (NCP). Thus, a positive map represents a bonafide state to state transformation if it is completely positive.

Under the action of a map $A : \rho_i \Rightarrow \rho_f = A(\rho_i)$, preservation of hermiticity i.e., $(\rho_f)_{s'r'}^* = (\rho_f)_{r's'}$ and the unit trace condition $\sum_{r'=1}^n (\rho_f)_{r'r'} = 1$ result in the following constraints on the elements of the $n^2 \times n^2$ process matrix A :

$$\begin{aligned} (\rho_f)_{r's'} &= (\rho_f)_{s'r'}^* \Rightarrow \sum_{r,s=1}^n A_{s'r';sr}^* (\rho_i)_{sr}^* = \sum_{r,s=1}^n A_{r's';rs} (\rho_i)_{rs} \\ &\Rightarrow \sum_{r,s=1}^n A_{s'r';sr}^* (\rho_i)_{rs} = \sum_{r,s=1}^n A_{r's';rs} (\rho_i)_{rs} \\ &\Rightarrow A_{r's';rs}^* = A_{s'r';sr} \end{aligned} \quad (8)$$

and

$$\begin{aligned} \sum_{r'=1}^n (\rho_f)_{r'r'} &= 1 \Rightarrow \sum_{r',r,s} A_{r'r';rs}^* (\rho_i)_{rs} \\ &\Rightarrow \sum_{r'=1}^n A_{r'r';rs} = \delta_{rs} \end{aligned} \quad (9)$$

where δ_{rs} denotes Kronecker delta symbol.

A realigned process matrix B was defined as [1, 2]

$$B_{r'r';s's} = A_{r's';sr}. \quad (10)$$

so that the hermiticity and unit trace conditions (8), (9) on the A -form can be expressed as

$$B_{r'r';s's} = B_{s's';r'r}^* \quad (11)$$

$$\sum_{r'=1}^n B_{r'r';r's} = \delta_{rs} \quad \Rightarrow \quad \text{Tr } B = n. \quad (12)$$

Thus, a physically valid A -form requires that the corresponding realigned matrix B (given by (10)) is a $n^2 \times n^2$ Hermitian matrix with trace n .

Furthermore, positivity of the density matrix $\rho_f = A(\rho_i) \geq 0$ leads to the following constraints on the elements of A and B respectively [1]:

$$\begin{aligned} \sum_{r,s,r',s'} x_r^* x_s A_{rs;r's'} y_{r'} y_{s'}^* &\geq 0, \\ \sum_{r,s,r',s'} x_r^* y_{r'} B_{rr';ss'} x_s y_{s'}^* &\geq 0. \end{aligned} \quad (13)$$

In other words, positivity $\rho_f = A(\rho_i) \geq 0$ of the density matrix requires that $B \geq 0$.

It is pertinent to point out that the B -form is represented by a Hermitian matrix whereas A is not; positivity of the B matrix highlights that the output density matrix is legitimate. For this reason Sudarshan, Mathews and Rau [1] highlighted that *the matrix B incorporates the kinematical restrictions on the dynamical law in a succinct fashion; we shall call B the dynamical matrix.* The A -form was used in Ref. [1] to define a linear map from input to output

density operators (where the elements of the input and output density matrices are arranged in the form of n^2 component columns). Beyond this initial definition, the A matrix was not recognized to have any clear role. Our focus here is to unravel the A -form to its full potential. We show in the next section that the A matrix introduced in the Sudarshan–Mathew–Rau paper exhibits an elegant canonical structure and it reveals itself as a powerful tool in capturing all the dynamical features reflected by the corresponding B -form [7].

3 Canonical structure of the A -form

Consider an orthonormal set

$$\{T_\mu, \mu = 0, 1, 2, \dots, n^2 - 1\}$$

of $n \times n$ matrices satisfying

$$\text{Tr}[T_\mu^\dagger T_\nu] = \delta_{\mu,\nu}. \quad (14)$$

We then construct a basis set

$$\{T_\mu \otimes T_\nu^*, \mu, \nu = 0, 1, \dots, n^2 - 1\} \quad (15)$$

of $n^2 \times n^2$ matrices and express the A matrix (see (3)) as follows:

$$A = \sum_{\mu,\nu=0}^{n^2-1} a_{\mu\nu} T_\mu \otimes T_\nu^* \quad (16)$$

where the expansion coefficients $a_{\mu\nu}$ are given by

$$a_{\mu\nu} = \text{Tr}[A(T_\mu^\dagger \otimes T_\nu^T)], \quad \mu, \nu = 0, 1, \dots, n^2 - 1. \quad (17)$$

The matrix elements $A_{r's';rs}$ of the A matrix are then given by

$$A_{r's';rs} = \sum_{\mu,\nu=0}^{n^2-1} a_{\mu\nu} [T_\mu]_{r'r} [T_\nu^*]_{s's}. \quad (18)$$

Let us examine the hermiticity preserving condition (8) on the expansion coefficients $a_{\alpha\beta}$:

$$A_{r's';rs}^* = A_{s'r';sr} \Rightarrow a_{\mu\nu}^* = a_{\nu\mu} \quad (19)$$

In other words, the coefficients $a_{\mu\nu}, \mu, \nu = 0, 1, \dots, n^2 - 1$ constitute a $n^2 \times n^2$ Hermitian matrix, which we denote by \mathcal{A} .

Let \mathcal{U} be a unitary matrix which diagonalizes \mathcal{A} , i.e.,

$$\mathcal{U} \mathcal{A} \mathcal{U}^\dagger = \mathcal{A}_0 = \begin{pmatrix} \lambda_0 & 0 & \dots & \dots & 0 \\ 0 & \lambda_1 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & \lambda_{n^2-1} \end{pmatrix} \quad (20)$$

where $\lambda_\mu, 0 \leq \mu \leq n^2 - 1$ denote the eigenvalues of \mathcal{A} . Thus we obtain

$$\begin{aligned} a_{\mu\nu} &= (\mathcal{U}^\dagger \mathcal{A}_0 \mathcal{U})_{\mu\nu} \\ &= \sum_\alpha \lambda_\alpha u_{\alpha\mu}^* u_{\alpha\nu}. \end{aligned} \quad (21)$$

Substituting (21) in (16) we obtain the following canonical structure of the A matrix:

$$\begin{aligned} A &= \sum_{\mu,\nu,\alpha} \lambda_\alpha u_{\alpha\mu}^* u_{\alpha\nu} (T_\mu \otimes T_\nu^*) \\ &= \sum_\alpha \lambda_\alpha (C_\alpha \otimes C_\alpha^*), \end{aligned} \quad (22)$$

where we have denoted

$$C_\alpha = \sum_{\mu=0}^{n^2-1} u_{\alpha\mu}^* T_\mu. \quad (23)$$

- Using (22) we can express the matrix elements of A as

$$A_{r's';rs} = \sum_{\alpha=0}^{n^2-1} \lambda_\alpha (C_\alpha)_{r'r} (C_\alpha^*)_{s's}. \quad (24)$$

Substituting (24) in (3) and simplifying, we obtain the following elegant structure for the action of the linear A -map on the column vector consisting of the elements of the input density matrix ρ_i :

$$\begin{aligned} (\rho_f)_{r's'} &= \sum_{r,s} A_{r's';rs} (\rho_i)_{rs} \\ &= \sum_{r,s,\alpha} \lambda_\alpha (C_\alpha)_{r'r} (C_\alpha^*)_{s's} (\rho_i)_{rs} \\ &= \sum_{r,s,\alpha} \lambda_\alpha (C_\alpha)_{r'r} (\rho_i)_{rs} (C_\alpha^\dagger)_{s's} \\ &= \sum_\alpha \lambda_\alpha (C_\alpha \rho_i C_\alpha^\dagger)_{r's'} \\ \Rightarrow \rho_f &= \sum_\mu \lambda_\mu C_\mu \rho_i C_\mu^\dagger. \end{aligned} \quad (25)$$

- From (25) the trace preservation condition (9) assumes the form

$$\begin{aligned} \text{Tr}(\rho_f) &= 1 \\ \Rightarrow \sum_\alpha \lambda_\alpha \text{Tr} (C_\alpha \rho_i C_\alpha^\dagger) &= \sum_\alpha \lambda_\alpha \text{Tr} (C_\alpha^\dagger C_\alpha \rho_i) = 1 \\ \Rightarrow \sum_\alpha \lambda_\alpha C_\alpha^\dagger C_\alpha &= I_n. \end{aligned} \quad (26)$$

- From (10) and (24) we may identify the elements of the realigned B matrix as

$$B_{r'r';s's} = \sum_\alpha \lambda_\alpha (C_\alpha)_{r'r} (C_\alpha^*)_{s's} \quad (27)$$

which happens to be the spectral decomposition of the dynamical B matrix with λ_μ being its eigenvalues.

Highlighting point here is that (27) brings out an explicit connection between the Hermitian (coefficient) matrix \mathcal{A} (see (16) and (19)) and the dynamical matrix B of Ref. [1]:

1. The eigenvalues of the coefficient matrix \mathcal{A} associated with the A -form are identically same as those of B .
2. A completely positive map requires that the coefficient matrix \mathcal{A} is positive (i.e., the eigenvalues λ_α are non-negative whenever the map is completely positive).
3. In the case of a completely positive map one may define a set $\{E_\alpha, \alpha = 0, 1, \dots, n^2-1\}$ of $n \times n$ matrices based on the canonical structure (22) of the A -map:

$$E_\alpha = \sqrt{\lambda_\alpha} C_\alpha. \quad (28)$$

Then the transformation $\rho_i \rightarrow \rho_f = A(\rho_i)$ gets expressed in terms of the Kraus operator-sum representation [8], i.e.,

$$\begin{aligned} (\rho_f)_{r's'} &= \sum_{r,s} A_{r's';rs} (\rho_i)_{rs} \\ &= \sum_{r,s,\alpha} \lambda_\alpha (E_\alpha)_{r'r} (E_\alpha^*)_{s's} (\rho_i)_{rs} \\ &= \sum_\alpha (E_\alpha)_{r'r} (\rho_i)_{rs} (E_\alpha^\dagger)_{s's} \\ &= \sum_\alpha (E_\alpha \rho_i E_\alpha^\dagger)_{r's'} \\ \Rightarrow \rho_f &= \sum_\alpha E_\alpha \rho_i E_\alpha^\dagger. \end{aligned} \quad (29)$$

We point out that the operator sum representation (29) was already described (via the spectral decomposition of the dynamical matrix B) by Sudarshan,

Mathews and Rau in their 1961 paper [1] and it was independently proposed by Kraus [8] after 10 years. The operators E_α (see (28),(29)) associated with a completely positive map are known as Kraus operators in the literature.

Summarizing, in this section we have shown that the canonical structure (22) of the A -form plays a significant role on its own—bringing forth all the required features of the quantum channel—without any necessity to invoke the realigned B -form. In the next section we employ the A -form to elucidate the completely positive or not completely positive behavior of some familiar qubit channels.

4 Canonical A-form of standard qubit maps

In this section we illustrate explicit 4×4 matrix forms of the canonical A -form and its equivalence with the dynamical matrix B of some standard qubit transformations.

4.1 Unitary map

Under the action of a unitary transformation we have

$$\rho_f = U \rho_i U^\dagger \implies (\rho_f)_{r's'} = \sum_{r,s} U_{r'r} U_{s's}^* (\rho_i)_{rs} = \sum_{r,s} (U \otimes U^*)_{r's';rs} (\rho_i)_{rs}. \quad (30)$$

We thus obtain (see (3))

$$(A_U)_{r's';rs} = (U \otimes U^*)_{r's';rs} \implies A_U = U \otimes U^* \quad (31)$$

Let us consider the 2×2 unitary matrix

$$\begin{aligned} U &= e^{i(\vec{\sigma} \cdot \hat{n})\theta/2}, \quad |\hat{n}|^2 = n_1^2 + n_2^2 + n_3^2 = 1 \\ &= I_2 \cos(\theta/2) + i \vec{\sigma} \cdot \hat{n} \sin(\theta/2) \\ &= \begin{pmatrix} \cos(\frac{\theta}{2}) + i n_3 \sin(\frac{\theta}{2}) & i n_- \sin(\frac{\theta}{2}) \\ i n_+ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) - i n_3 \sin(\frac{\theta}{2}) \end{pmatrix}, \quad n_\pm = (n_1 \pm i n_2). \end{aligned} \quad (32)$$

Then the matrix A_U (see (6), (31)) associated with the unitary matrix (32) takes the form

$$A_U = (I_2 \otimes I_2) \cos^2(\theta/2) + (\vec{\sigma} \cdot \hat{n} \otimes \vec{\sigma}^* \cdot \hat{n}) \sin^2(\theta/2) - i(I_2 \otimes \vec{\sigma}^* \cdot \hat{n} - \vec{\sigma} \cdot \hat{n} \otimes I_2) \cos(\theta/2) \sin(\theta/2). \quad (33)$$

Denoting $\sigma_0 = I_2$ and employing the basis set $\{T_\mu = \frac{1}{\sqrt{2}} \sigma_\mu\}$ of 2×2 matrices satisfying (see (14)) the conditions

$$\frac{1}{2} \text{Tr}[\sigma_\mu \sigma_\nu] = \delta_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,$$

we express (33) in the following compact form.

$$A_U = \frac{1}{2} \sum_{\mu,\nu=0,1,2,3} (a_U)_{\mu\nu} \sigma_\mu \otimes \sigma_\nu^*, \quad (34)$$

Observe that

$$\begin{aligned} (a_U)_{\mu\nu} &= \frac{1}{2} \text{Tr}[A_U (\sigma_\mu \otimes \sigma_\nu^*)] \\ &= \frac{1}{2} \text{Tr}[U \sigma_\mu] \text{Tr}[U \sigma_\nu]^* \end{aligned} \quad (35)$$

where $\text{Tr}[U \sigma_\mu], \mu = 0, 1, 2, 3$ is evaluated using (32):

$$\begin{aligned} \text{Tr}[U \sigma_0] &= 2 \cos\left(\frac{\theta}{2}\right), \\ \text{Tr}[U \sigma_k] &= 2i n_k \sin\left(\frac{\theta}{2}\right), \quad k = 1, 2, 3 \end{aligned} \quad (36)$$

Then the 4×4 coefficient matrix $\mathcal{A}_U = ((a_U)_{\mu\nu})$ (see (35)) associated with A_U is given by

$$\mathcal{A}_U = 2 X_U X_U^\dagger, \quad X_U = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ i n_1 \sin\left(\frac{\theta}{2}\right) \\ i n_2 \sin\left(\frac{\theta}{2}\right) \\ i n_3 \sin\left(\frac{\theta}{2}\right) \end{pmatrix}. \quad (37)$$

From (37) it is seen that the coefficient matrix \mathcal{A}_U is a rank-1 positive matrix with eigenvalue 2 and eigenvector X_U . The realigned B_U matrix matches exactly with the coefficient matrix \mathcal{A}_U , i.e.,

$$B_U = 2 X_U X_U^\dagger \equiv \mathcal{A}_U. \quad (38)$$

4.2 Pin map

Consider a linear A -form mapping every input state ρ_i to a fixed output state ρ_0 , i.e.,

$$A_{\text{pin}} : \rho_i \rightarrow \rho_0 = A_{\text{pin}}(\rho_i) \forall \rho_i. \quad (39)$$

Sudarshan, Mathews and Rau presented this map in terms of the B -form, which was termed as *relaxation generator*.

Here we would like to illustrate the canonical structure of the A -form associated with the qubit pin-map.

Let the fixed output density matrix of the qubit be given by

$$\rho_0 = \frac{1}{2}(I_2 + \vec{\sigma} \cdot \vec{p}_0). \quad (40)$$

The 4×4 matrix A_{pin} corresponding to the pin map is identified as follows:

$$\begin{aligned} \begin{pmatrix} 1 + (p_0)_3 \\ (p_0)_1 - i(p_0)_2 \\ (p_0)_1 + i(p_0)_2 \\ 1 - (p_0)_3 \end{pmatrix} &= \begin{pmatrix} (A_{\text{pin}})_{00;00} & (A_{\text{pin}})_{00;01} & (A_{\text{pin}})_{00;10} & (A_{\text{pin}})_{00;11} \\ (A_{\text{pin}})_{01;00} & (A_{\text{pin}})_{01;01} & (A_{\text{pin}})_{01;10} & (A_{\text{pin}})_{01;11} \\ (A_{\text{pin}})_{10;00} & (A_{\text{pin}})_{10;01} & (A_{\text{pin}})_{10;10} & (A_{\text{pin}})_{10;11} \\ (A_{\text{pin}})_{11;00} & (A_{\text{pin}})_{11;01} & (A_{\text{pin}})_{11;10} & (A_{\text{pin}})_{11;11} \end{pmatrix} \begin{pmatrix} 1 + (p_i)_3 \\ (p_i)_1 - i(p_i)_2 \\ (p_i)_1 + i(p_i)_2 \\ 1 - (p_i)_3 \end{pmatrix} \forall \vec{p}_i \\ \Rightarrow A_{\text{pin}} &= \frac{1}{2} \begin{pmatrix} 1 + (p_0)_3 & 0 & 0 & 1 + (p_0)_3 \\ (p_0)_1 - i(p_0)_2 & 0 & 0 & (p_0)_1 - i(p_0)_2 \\ (p_0)_1 + i(p_0)_2 & 0 & 0 & (p_0)_1 + i(p_0)_2 \\ 1 - (p_0)_3 & 0 & 0 & 1 - (p_0)_3 \end{pmatrix}. \end{aligned} \quad (41)$$

Using the orthonormal basis set of matrices $\{\frac{\sigma_\mu}{\sqrt{2}}, \mu = 0, 1, 2, 3\}$ we expand

$$A_{\text{pin}} = \frac{1}{2} \sum_{\mu, \nu} (a_{\text{pin}})_{\mu\nu} \sigma_\mu \otimes \sigma_\nu^*, \text{ where } (a_{\text{pin}})_{\mu\nu} = \frac{1}{2} \text{Tr}[A_{\text{pin}} (\sigma_\mu \otimes \sigma_\nu^*)].$$

The coefficient matrix $\mathcal{A}_{\text{pin}} = ((a_{\text{pin}})_{\mu\nu})$ is then found to be

$$\mathcal{A}_{\text{pin}} = \frac{1}{2} \begin{pmatrix} 1 & (p_0)_1 & (p_0)_2 & (p_0)_3 \\ (p_0)_1 & 1 & -i(p_0)_3 & i(p_0)_2 \\ (p_0)_2 & i(p_0)_3 & 1 & -i(p_0)_1 \\ (p_0)_3 & -i(p_0)_2 & i(p_0)_1 & 1 \end{pmatrix}. \quad (42)$$

The eigenvalues of \mathcal{A}_{pin} are given by

$$\lambda_0 = \lambda_1 = \frac{1}{2}(1 + |\vec{p}_0|), \quad \lambda_2 = \lambda_3 = \frac{1}{2}(1 - |\vec{p}_0|). \quad (43)$$

Clearly, the eigenvalues of \mathcal{A}_{pin} are all positive ensuring that the pin map is completely positive.

The B matrix associated with the pin map is constructed using the explicit matrix form of A_{pin} (see (41)):

$$\begin{aligned} B_{\text{pin}} &= \frac{1}{2} \begin{pmatrix} 1 + (p_0)_3 & 0 & (p_0)_1 - i(p_0)_2 & 0 \\ 0 & 1 + (p_0)_3 & 0 & (p_0)_1 - i(p_0)_2 \\ (p_0)_1 + i(p_0)_2 & 0 & 1 - (p_0)_3 & 0 \\ 0 & (p_0)_1 + i(p_0)_2 & 0 & 1 - (p_0)_3 \end{pmatrix} \\ &= \rho_0 \otimes I_2 \end{aligned} \quad (44)$$

Eigenvalues of B_{pin} match with those of the coefficient matrix \mathcal{A}_{pin} (see (43)), thus establishing the equivalence between the two.

4.3 Transpose map

Consider the transpose map $A_T : \rho \rightarrow \rho^T$ on qubit density matrices (see (4)). We obtain the associated 4×4 matrix form of A_T as

$$A_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (45)$$

Employing the basis set $\{\frac{\sigma_\mu}{\sqrt{2}}, \mu = 0, 1, 2, 3\}$ we express

$$A_T = \frac{1}{2} \sum_{\mu, \nu=0}^3 (a_T)_{\mu\nu} (\sigma_\mu \otimes \sigma_\nu^*) \quad (46)$$

to obtain the following explicit structure for the coefficient matrix \mathcal{A}_T :

$$\mathcal{A}_T = \text{diag}(-1, 1, 1, 1). \quad (47)$$

The matrix \mathcal{A}_T is not positive (one of the eigenvalues of \mathcal{A}_T is -1) which points towards the not-completely positive nature of the transpose map.

From the explicit matrix structure of A_T (see (45)) it is easy to see that the realigned dynamical matrix $B_T \equiv A_T$. The eigenvalues of B_T match with those of the coefficient matrix \mathcal{A}_T (see (47)).

4.4 Projection of the Bloch sphere onto its equatorial plane

A map that projects the entire Bloch sphere onto the equatorial plane is defined by the transformation of the Bloch vector

$$(p_1, p_2, p_3) \rightarrow (p_1, p_2, 0).$$

This leads to the following linear transformation

$$\begin{pmatrix} 1 \\ p_1 - i p_2 \\ p_1 + i p_2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + p_3 \\ p_1 - i p_2 \\ p_1 + i p_2 \\ 1 - p_3 \end{pmatrix}. \quad (48)$$

We then express $A_P = \frac{1}{2} \sum_{\mu, \nu} (a_P)_{\mu\nu} (\sigma_\mu \otimes \sigma_\nu^*)$ to obtain

$$\mathcal{A}_P = \frac{1}{2} \text{diag}(3, 1, 1, -1). \quad (49)$$

Negative eigenvalues of \mathcal{A}_P clearly indicate that projection of the Bloch sphere onto the equatorial plane is not physical as it corresponds to a not completely positive

map. The dynamical matrix B_P is then obtained using the realignment $(B_P)_{r'r';s's} = (A_P)_{r's';r's}$:

$$B_P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}. \quad (50)$$

Eigenvalues of B_P are same as those of \mathcal{A}_P .

4.5 Bit flip channel

A qubit bit flip channel reverses the state of a qubit from $|0\rangle$ to $|1\rangle$ with probability $1 - p$, $0 \leq p \leq 1$; the channel keeps the states unaltered with a probability p . This is a completely positive map equipped with the Kraus operators given by [3]

$$E_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 \equiv \sqrt{1-p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (51)$$

Using the operator-sum representation $\rho_f = E_0 \rho_i E_0^\dagger + E_1 \rho_i E_1^\dagger$ we construct the associated A matrix:

$$A_{\text{BF}} = \begin{pmatrix} p & 0 & 0 & 1-p \\ 0 & p & 1-p & 0 \\ 0 & 1-p & p & 0 \\ 1-p & 0 & 0 & p \end{pmatrix} \quad (52)$$

Adopting the matrix basis $\{\frac{\sigma_\mu}{\sqrt{2}}, \mu = 0, 1, 2, 3\}$, as in all other examples studied earlier, we compute the coefficient matrix $\mathcal{A}_{\text{BF}} = (a_{\mu\nu})$ associated with A_{BF} :

$$\mathcal{A}_{\text{BF}} = \begin{pmatrix} 2p & 0 & 0 & 0 \\ 0 & 2(1-p) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (53)$$

Note that the eigenvalues $2p, 2(1-p)$ of \mathcal{A}_{BF} are positive and ascertain the completely positive nature of the bit flip channel.

4.6 Phase flip channel

A qubit phase flip channel is equipped with the Kraus operators [3]

$$E_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (54)$$

The linear map A_{PF} associated with the phase flip channel is given explicitly as a 4×4 matrix form:

$$A_{\text{PF}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2p-1 & 0 & 0 \\ 0 & 0 & 2p-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (55)$$

Then the associated coefficient matrix \mathcal{A}_{PF} is found to be

$$\mathcal{A}_{\text{PF}} = \begin{pmatrix} 2p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1-p) \end{pmatrix} \quad (56)$$

The eigenvalues $2p, 2(1-p)$ of \mathcal{A}_{PF} are necessarily positive and confirm the legitimacy (complete positivity) of the phase flip channel.

5 Summary

The dynamical A and B -maps were pioneered by Sudarshan, Mathews and Rau [1] in the context of open system dynamics. Unfortunately this seminal 1961 paper did not receive its due attention in the field, although it contained all the details of finite dimensional quantum channels. In this work we have elaborated on the canonical structure of the A -form, establishing that it offers an alternative approach to recognize the completely positive/not completely positive nature of quantum channels. We have illustrated the canonical structure of the A map in several standard examples of qubit maps. A new geometrical representation based on the Lorentz singular value decomposition [9] of the canonical A -form associated with qubit transformations is being prepared and it will be presented separately as a sequel to the present work.

Dedication

We dedicate this tutorial article as a mark of our reverence to Professor E. C. G. Sudarshan.

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