A Class of Stochastic and Distributions-Free Quantum Mechanical Evolution Equations

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A procedure allowing to construct rigorously discrete as well as continuum deterministic evolution equations from stochastic evolution equations is developed using Dirac's bra–ket notation. This procedure is an extension of an approach previously used by the author coined Discrete Stochastic Evolution Equations. Definitions and examples of discrete as well as continuum one-dimensional lattices are developed in detail in order to show the basic tools that allow to construct Schrödinger-like equations. Extension to multi-dimensional lattices are studied in order to provide a wider exposition and the one-dimensional cases are derived as special cases, as expected. Some variants of the procedure allow the construction of other evolution equations. Also, using a limiting procedure, it is possible to derive the Schrödinger equation from the Schrödinger-like equations. Another possible approach is given in the appendix.

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1 Introduction

In this paper, a deterministic quantum mechanical evolution equation is derived from a set of stochastic quantum mechanical evolution equations using Dirac's bra–ket notation, which can be considered an extension of an approach coined Discrete Stochastic Evolution Equations (DSEE) [1]. Some illustrative examples that show the versatility of this approach can be found in [1–9]. Within this context, discrete as well as continuum Schrödinger-like equations will be obtained without the use of distributions like Dirac's delta. This goal is achieved using an appropriate split of the discrete as well as the continuum Hamiltonian, allowing a complete rigorous derivation of the usual equation proposed or obtained in the literature, e.g. [9–11]. Lot of work was done and is being done on the subject of Quantum Stochastic Processes, see for example the classical books [12, 13] or more recently from a more mathematical grounds, e.g. [14–17]. Appropriate expansions in a finite centered differences series as well as in a Taylor series for the discrete and continuum evolution equation, respectively, allows to complete the basic tools used for deriving distribution-free equations in a straightforward way. The possibility of obtaining deterministic evolution equations from stochastic evolution equations is based on the assumption that the Hamiltonian and the wave functions are statistically independent. This means that the averages over realization of products of the Hamiltonian with the wave functions factorizes. One more assumption that allows the obtention of discrete as well as continuum Schrödinger-like equations is that the Hamiltonian can be split in a sum of two terms, as will be seen in all the examples given below. An explanation about the use of Schrödinger-like equations instead of Schrödinger equation is in order. It is well known (see e.g. [11, §16]) that the Hamiltonian allowing to obtain a deterministic one-dimensional Schrödinger equation is expressed in terms of distributions like

$$\begin{aligned} H(x, x') &= -\frac{\hbar}{2m}\delta''(x-x') + V(x)\delta(x-x') \\ &= \left(-\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)\right)\delta(x-x'), \end{aligned}$$

where x and x' are two points, $\delta''(x-x')$ is the second derivative of the delta function $\delta(x-x')$, $V(x)$ is the potential, m is the mass of the particle, and $\hbar$ is the reduced Plank constant. On the other hand, as *Schrödinger-like equation* will be referred to any equation with the same form of the Schrödinger equation when the Hamiltonian and the wave function are expanded in a Taylor series up to an appropriate order in $\triangle x' = x' - x$, as we will see below. The question that will be answered is: Is it possible to find a Hamiltonian, independent of distributions, that after introducing it in the general evolution equations

$$\frac{d\psi_i(t)}{dt} = -\frac{\imath}{\hbar}\sum_{i'} H_{i,i'}(t)\psi_{i'}(t), \qquad \forall i, i' \in \Lambda_1,$$

or

$$\frac{\partial\psi(x,t)}{\partial t} = -\frac{\imath}{\hbar}\int H(x,x',t)\psi(x',t)dx', \qquad \forall x, x' \in \Re,$$

generates a discrete or continuum Schrödinger-like equation, respectively? Note that the Hamiltonian can also be time dependent as it will be seen below. The answer is yes for both discrete as well as continuum cases. The Schrödinger equation can be obtained using a limiting procedure. To this end a discrete as well as continuum Taylor series expansion (e.g. [18]) like

$$H(x,x',t) = H(x, x+\Delta x', t) = \lim_{h\to 0^+}\sum_{m=0}^{\infty}\frac{\Delta_h^m H(x,x,t)}{h^m}\frac{\Delta x'^m}{m!},$$

$$\psi(x',t) = \psi(x+\Delta x', t) = \lim_{h\to 0^+}\sum_{n=0}^{\infty}\frac{\Delta_h^n \psi(x,t)}{h^n}\frac{\Delta x'^n}{n!},$$

given the discrete case with h finite

$$H_{i,i'}(t) = H_{i,i+\Delta i'}(t) = \sum_{m=0}^{\infty}\frac{\Delta^m H_{i,i}(t)}{\Delta i^m}\frac{\Delta i'^m}{m!},$$

$$\psi_{i'}(t) = \psi_{i+\Delta i'}(t) = \sum_{n=0}^{\infty}\frac{\Delta^n \psi_i(t)}{\Delta i^n}\frac{\Delta i'^n}{n!},$$

or the continuum case when $h \to 0^+$, given the usual Taylor series expansion

$$H(x,x',t) = H(x, x+\Delta x', t) = \sum_{m=0}^{\infty}\frac{\partial^m H(x,x,t)}{\partial x^m}\frac{\Delta x'^m}{m!},$$

$$\psi(x',t) = \psi(x+\Delta x', t) = \sum_{n=0}^{\infty}\frac{\partial^n \psi(x,t)}{\partial x^n}\frac{\Delta x'^n}{n!}.$$

The paper is organized as follows. In Section 2, the simplest case is considered and the basic definitions of a stochastic one-dimensional discrete evolution equation of a ket is used in order to find a discrete deterministic evolution equation for the wave function and the corresponding discrete Schrödinger-like equations. This approach allows to prove that the usual proposed evolution equations can be obtained rigorously from first principles. In Section 3, the basic definitions of a stochastic one-dimensional evolution equation of a ket is used in order to find a continuum deterministic evolution equation for the wave function and the corresponding continuum Schrödinger-like equation. In Section 4, the extension to d-dimensional lattices is considered and a continuum Schrödinger-like equation is obtained. The one-dimensional case is obtained as a special case, as expected. In Section 5, some additional illustrative examples are considered in order to provide a better understanding of the procedures. In Section 6, it is proved that the Schrödinger equation can be obtained as a limiting case of Schrödinger-like equations. In Section 7, conclusions, some generalizations, and perspectives are given. Finally, another possible approach is given in the Appendix.

2 The discrete one-dimensional lattice

Beginning, for the sake of simplicity in the presentation, with the simplest stochastic evolution of a one-dimensional ket, namely

$$\begin{aligned} |\,\psi^{(r)}(t+\Delta t)\rangle &= U^{(r)}(t+\Delta t, t)\,|\,\psi^{(r)}(t)\rangle \\ &= \left(1 - \frac{\imath}{\hbar}H'^{(r)}(t)\Delta t + O(\Delta t^2)\right)|\,\psi^{(r)}(t)\rangle \\ &= |\,\psi^{(r)}(t)\rangle - \frac{\imath}{\hbar}H'^{(r)}(t)\Delta t\,|\,\psi^{(r)}(t)\rangle, \quad (1) \end{aligned}$$

where (r) indicates that the evolution refers to a particular realization and a "discrete" Taylor series expansion of $U^{(r)}(t+\Delta t, t)$ up to $O(\Delta t)$ and $U^{(r)}(t,t) = 1$, was used. In this way, an evolution equation similar to the one for the dynamical variables given in [9] is obtained for the corresponding ket $| \psi^{(r)}(t)\rangle$. The so called "weights" in [9], in this case, are $\frac{\Delta U^{(r)}(t,t)}{\Delta t}\Delta t = -\frac{\imath}{\hbar}H'^{(r)}(t)\Delta t$, where $\imath = \sqrt{-1}$, $\hbar$ is the reduced Plank constant and $H'^{(r)}(t)$ is the "primed Hamiltonian" which will be the Hamiltonian when $\Delta t \to 0$. As usual, if the evolution equations for the probability amplitudes are required, the following operations must be done on both sides of Eq. (1): 1) multiplying by a bra $\langle i|$, 2) summing $-\langle i \mid \psi^{(r)}(t)\rangle$, 3) dividing by Δt , and 4) letting $\Delta t \to 0$. After these steps are completed, $\frac{dU^{(r)}(t,t)}{dt} = -\frac{\imath}{\hbar}H^{(r)}(t)$ and the following evolution equation is obtained

$$\begin{aligned}\frac{d\psi_i^{(r)}(t)}{dt} &= \langle i| - \frac{\imath}{\hbar}H^{(r)}(t) \mid \psi^{(r)}(t)\rangle \\ &= -\frac{\imath}{\hbar}\langle i|H^{(r)}(t)\sum_{i'} \mid i'\rangle\langle i' \mid \psi^{(r)}(t)\rangle \\ &= -\frac{\imath}{\hbar}\sum_{i'}\langle i|H^{(r)}(t) \mid i'\rangle\langle i' \mid \psi^{(r)}(t)\rangle \\ &= -\frac{\imath}{\hbar}\sum_{i'} H_{i,i'}^{(r)}(t)\psi_i'^{(r)}(t), \quad \forall i, i' \in \Lambda_1, \quad (2)\end{aligned}$$

where Λ_1 is the set of sites of the lattice with periodic boundary conditions, and

$$\begin{aligned}\frac{d\psi_i^{(r)}(t)}{dt} &= \lim_{\Delta t\to 0}\frac{\psi_i^{(r)}(t+\Delta t) - \psi_i^{(r)}(t)}{\Delta t}, \\ | \psi^{(r)}(t)\rangle &= \sum_{i'} \mid i'\rangle\langle i' \mid \psi^{(r)}(t)\rangle, \\ H_{i,i'}^{(r)}(t) &= \langle i|H^{(r)}(t) \mid i'\rangle, \\ \psi_i^{(r)}(t) &= \langle i \mid \psi^{(r)}(t)\rangle, \\ \psi_{i'}^{(r)}(t) &= \langle i' \mid \psi^{(r)}(t)\rangle,\end{aligned}$$

was used. Note that it was used i, i' as integer number but these discrete sites are particular cases of $a_1 i$ and $a_1 i'$ corresponding to the coordinates of the lattice sites, which are equal only if the spacing of the lattice a_1 is one. It must be emphasized that sometimes in the literature [9, 11] the notation used is $\langle i \mid \psi^{(r)}(t)\rangle = C_i^{(r)}(t)$ instead of $\psi_i^{(r)}(t)$, but here we prefer to use the more common notation. If a deterministic evolution equation is required, an average over realization on both sides of Eq. (2) is needed. The final result is

$$\frac{d\psi_i(t)}{dt} = -\frac{\imath}{\hbar}\sum_{i'} H_{i,i'}(t)\psi_{i'}(t), \qquad \forall i, i' \in \Lambda_1, \quad (3)$$

where,

$$\begin{aligned}\psi_i(t) &= \overline{\psi_i^{(r)}(t)}, \\ H_{i,i'}(t) &= \overline{H_{i,i'}^{(r)}(t)}, \\ H_{i,i'}(t)\psi_{i'}(t) &= \overline{H_{i,i'}^{(r)}(t)\psi_{i'}^{(r)}(t)} = \overline{H_{i,i'}^{(r)}(t)}\ \ \overline{\psi_{i'}^{(r)}(t)}. \quad (4)\end{aligned}$$

It is easy to see that a factorization was assumed, which means that $H_{i,i'}^{(r)}(t)$ and $\psi_{i'}^{(r)}(t)$ are statistically independent. In other words, a "discrete" deterministic evolution equation as the one given in Eq. (3) can only be obtained if the stochastic Hamiltonian and the probability amplitudes are statistically independent. One way different to the ones considered in [9, 11], where a discrete Schrödinger-like equation is obtained, is by making $i' = i + \triangle i'$ with $\triangle i' = i' - i$, and letting to split the Hamiltonian as $H_{i,i'}(t) = (H_1)_{i,i'} + (H_0)_{ii}\frac{\psi_i(t)}{\psi_{i'}(t)}$, where $(H_1)_{i,i'}$ and $(H_0)_{ii}$ are both time independent. Using these definitions, Eq. (3) becomes

$$\begin{aligned}\frac{d\psi_i(t)}{dt} &= -\frac{\imath}{\hbar}\left(\sum_{i'}(H_1)_{i,i'}\psi_{i'}(t) + \sum_{i'}(H_0)_{i,i}\psi_i(t)\right) \\ &= -\frac{\imath}{\hbar}(S_1 + S_0), \quad (5)\end{aligned}$$

where

$$\begin{aligned}S_1 &= \sum_{i'}(H_1)_{i,i'}\psi_{i'}(t) = \sum_{i'}(H_1)_{i,i+\triangle i'}\psi_{i+\triangle i'}(t) \\ &\approx \sum_{i'}(H_1)_{i,i}\left(\psi_i(t) + \frac{\triangle\psi_i(t)}{\triangle i}(\triangle i') + \frac{1}{2}\frac{\triangle^2\psi_i(t)}{\triangle i^2}(\triangle i')^2 + \cdots\right), \\ &= (H_1)_{i,i}\sum_{i'}\left(\psi_i(t) + \frac{\triangle\psi_i(t)}{\triangle i}(\triangle i') + \frac{1}{2}\frac{\triangle^2\psi_i(t)}{\triangle i^2}(\triangle i')^2 + \cdots\right) \\ &= (H_1)_{i,i}\left(\psi_i(t)s_0 + \frac{\triangle\psi_i(t)}{\triangle i}s_1 + \frac{1}{2}\frac{\triangle^2\psi_i(t)}{\triangle i^2}s_2\right), \\ S_0 &= \sum_{i'}(H_0)_{i,i}\psi_i(t) = (H_0)_{i,i}\psi_i(t)\sum_{i'}(1) = (H_0)_{i,i}\psi_i(t)s_0. \quad (6)\end{aligned}$$

In S_1 it was used

$$(H_1)_{i,i+\triangle i'} = (H_1)_{i,i} + 0(\triangle i'),$$

$$\psi_{i+\triangle i'}(t) = \psi_i(t) + \frac{\triangle\psi_i(t)}{\triangle i}(\triangle i') + \frac{1}{2}\frac{\triangle^2\psi_i(t)}{\triangle i^2}(\triangle i')^2 + O((\triangle i')^3),$$

where, the finite central (or centered) differences used are $\triangle i = a_1$, $\triangle i^2 = a_1^2$, $\triangle\psi_i(t) = \psi_{i+\frac{1}{2}}(t) - \psi_{i-\frac{1}{2}}(t)$, and $\triangle^2\psi_i(t) = \triangle(\triangle\psi_i(t)) = \psi_{i-1}(t) - 2\psi_i(t) + \psi_{i+1}(t)$.

The summations s_0, s_1, and s_2 are

$$\begin{aligned}s_0 &= \sum_j(1) = \overbrace{1 + \cdots + 1}^{N_1\ \text{times}} = N_1 \approx 2N_1', \\ s_1 &= \sum_j(\triangle j) = \left(-N_1' + \cdots + 0 + \cdots + N_1'\right) = 0, \\ s_2 &= \sum_j(\triangle j)^2 = \left(-N_1'\right)^2 + \cdots + (0)^2 + \cdots + \left(N_1'\right)^2 \\ &= \frac{2}{3}\left(N_1'\right)^3 + \left(N_1'\right)^2 + \frac{1}{3}\left(N_1'\right) \approx \frac{2}{3}\left(N_1'\right)^3, \quad (7)\end{aligned}$$

where, if the number of lattice sites N_1 is odd and $N_1 \gg 1$, then $N_1' = \frac{N_1-1}{2} \approx \frac{N_1}{2}$. Note that in Eq. (5), using this approximation, the time derivative of $\psi_i(t)$ depends only on the nearest neighbor values like a continuum Schrödinger-like equation where only second order partial derivative with respect to the spatial coordinate appears in the evolution equation, as will be shown below. Using the results of Eqs. (6) and (7) in Eq. (5), it is easy to find

$$\begin{aligned}\frac{d\psi_i(t)}{dt} &= -\frac{\imath}{\hbar}\left(s_2\frac{(H_1)_{i,i}}{2a_1^2}\left[\psi_{i-1}(t)-2\psi_i(t)+\psi_{i+1}(t)\right]\right)\\ &\quad -\frac{\imath}{\hbar}s_0\left[(H_0)_{i,i}+(H_1)_{ii}\right]\psi_i(t)\\ &= -\frac{\imath}{\hbar}s_2\frac{(H_1)_{i,i}}{2a_1^2}\left[\psi_{i-1}(t)+\psi_{i+1}(t)\right]\\ &\quad -\frac{\imath}{\hbar}(\triangle H)_{i,i}\psi_i(t), \qquad \forall i\in\Lambda_1, \quad (8)\end{aligned}$$

where $(\triangle H)_{i,i} = s_0[(H_0)_{i,i}+(H_1)_{i,i}]-2\frac{(H_1)_{i,i}}{2a_1^2}s_2 = (H_0)_{i,i}s_0+(H_1)_{i,i}\left(s_0-\frac{s_2}{a_1^2}\right)$. Note that this discrete evolution equation is the same to the one proposed in [11], if $(H_1)_{i,i} = -\frac{2a_1^2 A}{s_2}$ and $(H_0)_{i,i} = \frac{2a_1^2 A s_0+s_2E_0-2As_2}{s_0 s_2}$. These values are obtained after equating each of the coefficients with the ones given in [11] and solving the set of linear equations

$$\begin{aligned}(H_0)_{i,i}s_0+(H_1)_{i,i}\left(s_0-\frac{s_2}{a_1^2}\right) &= E_0,\\ s_2\frac{(H_1)_{i,i}}{2a_1^2} &= -A,\end{aligned}$$

for $(H_1)_{i,i}$ and $(H_0)_{i,i}$.

The meaning of the coefficients in [11] are: E_0 is a constant that allows to choose the zero of the energy, A is a constant independent of t, and $b = a_1$ is the spacing of the lattice. Of course, here Eq. (8) was derived from first principles.

3 The continuum one-dimensional lattice

The way of obtaining a continuum Schrödinger-like equation is to transform Eqs. (2) and (3) making the transformations

$$\begin{aligned}\psi_i^{(r)}(t) &\to \psi^{(r)}(x,t) = \langle x\mid\psi^{(r)}(t)\rangle,\\ \psi_{i'}^{(r)}(t) &\to \psi^{(r)}(x',t) = \langle x'\mid\psi^{(r)}(t)\rangle,\\ H_{i,i'}^{(r)}(t) &\to H^{(r)}(x,x',t) = \langle x\mid H^{(r)}(t)\mid x'\rangle,\end{aligned}$$

where the discrete indices i and i' where replaced by the continuous variables x and x', respectively. After transforming the summation into an integral like

$$-\frac{\imath}{\hbar}\sum_{i'}H_{i,i'}^{(r)}(t)\psi_{i'}^{(r)}(t) \to -\frac{\imath}{\hbar}\int H^{(r)}(x,x',t)\psi^{(r)}(x',t)dx',$$

where dx' is the differential length, $H^{(r)}(x,x',t)$ is the Hamiltonian density or the Hamiltonian per unit of length (or, in general, d-dimensional volume), and $\psi^{(r)}(x,t)$ is the wave function. *The reader should be warned that even when the same symbol was used for the discrete and continuum Hamiltonian, they are not the same. Also hereafter Hamiltonian will be used for both the discrete and continuum case*. Moreover, in a d-dimensional lattice, the differential volume is $dV_d' = dx_1'...dx_d'$, and the integral, as usual, is a multiple integral, one per dimension as it will be shown below. The final result is

$$\frac{\partial\psi^{(r)}(x,t)}{\partial t} = -\frac{\imath}{\hbar}\int H^{(r)}(x,x',t)\psi^{(r)}(x',t)dx', \qquad \forall x,x'\in\Re, (9)$$

and after an average over realization

$$\frac{\partial\psi(x,t)}{\partial t} = -\frac{\imath}{\hbar}\int H(x,x',t)\psi(x',t)dx' \qquad \forall x,x'\in\Re, (10)$$

where it was assumed that $H^{(r)}(x,x',t)$ and $\psi^{(r)}(x',t)$ are statistically independent, consequently, $\overline{H^{(r)}(x,x',t)\psi^{(r)}(x',t)} = \overline{H^{(r)}(x,x',t)}\ \overline{\psi^{(r)}(x',t)} = H(x,x',t)\psi(x',t)$. If it is needed to obtain a Schrödinger-like equation, the Hamiltonian must be split in a sum like $H(x,x',t) = H_1(x,x') + H_0(x,x)\frac{\psi(x,t)}{\psi(x',t)}$, where $H_0(x,x)$ and $H_1(x,x')$ are both time independent. Then Eq. (10) becomes

$$\begin{aligned}\frac{\partial\psi(x,t)}{\partial t} &= -\frac{\imath}{\hbar}\int H_1(x,x')\psi(x',t)dx'\\ &\quad -\frac{\imath}{\hbar}I_1H_0(x,x)\psi(x,t), \qquad (11)\end{aligned}$$

where, making $dx' = d(\triangle x')$ with $\triangle x' = x'-x$ [see Eqs. (14) and (15) below],

$$I_1 = \int_{-L_1'}^{L_1'} d(\triangle x') = 2L_1'.$$

In order to obtain a Taylor series expansion of the right-hand side of Eq. (11), it is necessary to rewrite the evolution equation in a convenient way

$$\begin{aligned}\psi(x',t) &= \psi(x+\triangle x',t) = \psi(x,t)+\frac{\partial\psi(x,t)}{\partial x}(\triangle x')\\ &\quad +\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}(\triangle x')^2+O((\triangle x')^3)\\ H_1(x,x') &= H_1(x,x+\triangle x') = H_1(x,x)+O(\triangle x').\end{aligned} \qquad (12)$$

After replacing Eq. (12) in Eq. (11) and integrating over x' it is easily found

$$\frac{\partial\psi(x,t)}{\partial t} = -\frac{\imath}{\hbar}(h_1+h_2+h_3+h_0)+O((\triangle x')^3), \quad (13)$$

where

$$\begin{aligned} h_1 &= H_1(x,x)\psi(x,t)I_1 \\ h_2 &= H_1(x,x)\frac{\partial\psi(x,t)}{\partial x}I_2 \\ h_3 &= H_1(x,x)\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}I_3 \\ h_0 &= I_1 H_0(x,x)\psi(x,t), \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_{x-L_1'}^{x+L_1'} dx' = \int_{-L_1'}^{L_1'} d(\triangle x') = 2L_1', \\ I_2 &= \int_{-L_1'}^{L_1'} (\triangle x')d(\triangle x') = \left[\frac{(\triangle x')^2}{2}\right]_{-L_1'}^{L_1'} = 0, \\ I_3 &= \int_{-L_1'}^{L_1'} (\triangle x')^2 d(\triangle x') = \left[\frac{(\triangle x')^3}{3}\right]_{-L_1'}^{L_1'} = 2\left[\frac{(L_1')^3}{3}\right]. \end{aligned} \tag{14}$$

Note that these integrals are the same as the summations s_0, s_1, and s_2, in Eq. (7), except for the fact that half the number of points of the lattice N_1 is here half the length of the lattice L_1. In Eq. (14) it was used the following change of variable

$$d(\triangle x') = \frac{d(x'-x)}{dx'}dx' = dx'. \tag{15}$$

Letting

$$\begin{aligned} \left[H_0(x,x) + H_1(x,x)\right] I_1 &= V(x), \\ \frac{1}{2}H_1(x,x)I_3 &= \left(\frac{-\hbar^2}{2m_{\text{eff}}}\right), \end{aligned} \tag{16}$$

in Eq. (13), the usual Schrödinger-like equation is obtained. Note that it is possible to choose $H_0(x,x)$, $H_1(x,x)$, and L_1, in such a way that the results obtained from a discrete lattice, given in [9,11], are recovered. The final result is

$$\begin{aligned} H_0(x,x) &= \frac{E_0}{I_1}, \\ H_1(x,x) &= -\frac{2A}{I_3}, \\ L_1' &= b\sqrt{3}. \end{aligned} \tag{17}$$

It is not difficult to see that, even when the formal results obtained in Eq. (16) are correct, in Eq. (17) the value of half the length of the lattice, obtained after equating $m_{\text{eff}} = \left(\frac{\hbar^2}{2Ab^2}\right)$, given in [11], to $m_{\text{eff}} = \left(\frac{-\hbar^2}{H_1(x,x)I_3}\right)$, is too small and consequently a meaningless result. However, we could provide a "reasonable physical assumption" that allows to obtain an alternative appropriate result: In order to achieve this goal, let us assume that the Hamiltonian is different from zero only inside an interval $-L_1'' \leq \triangle x' \leq L_1''$ with $|L_1''| < L_1$, then the integrals in Eq. (14) are

$$\begin{aligned} I_1' &= \int_{x-L_1''}^{x+L_1''} dx' = \int_{-L_1''}^{L_1''} d(\triangle x') = 2L_1'', \\ I_2' &= \int_{-L_1''}^{L_1''} (\triangle x')d(\triangle x') = \left[\frac{(\triangle x')^2}{2}\right]_{-L_1''}^{L_1''} = 0, \\ I_3' &= \int_{-L_1''}^{L_1''} (\triangle x')^2 d(\triangle x') = \left[\frac{(\triangle x')^3}{3}\right]_{-L_1''}^{L_1''} = 2\left[\frac{(L_1'')^3}{3}\right], \end{aligned} \tag{18}$$

and consequently, Eq. (16) becomes

$$\begin{aligned} \left[H_0'(x,x) + H_1'(x,x)\right] I_1' &= V(x), \\ \frac{1}{2}H_1'(x,x)I_3' &= \left(\frac{-\hbar^2}{2m_{\text{eff}}}\right), \end{aligned} \tag{19}$$

and Eq. (17) must be changed to

$$\begin{aligned} H_0'(x,x) &= \frac{E_0}{I_1'}, \\ H_1'(x,x) &= -\frac{2A}{I_3'}, \\ L_1'' &= b\sqrt{3}, \end{aligned} \tag{20}$$

where L_1'' was obtained after equating $m_{\text{eff}} = \left(\frac{\hbar^2}{2Ab^2}\right)$ to $m_{\text{eff}} = \left(\frac{-\hbar^2}{H_1'(x,x)I_3'}\right)$. This sort of "cut off" provides an interval which is much smaller than the length of the lattice L_1 and is very close to the lattice spacing $b = a_1$, which is a very reasonable physical result and justifies the approximate value of the Hamiltonian in Eq. (12).

4 Extensions to multi-dimensional lattice

The extension to a d-dimensional lattice is straightforward. Defining d-dimensional vectors $\overrightarrow{x}$ and $\overrightarrow{x}'$, the Taylor series expansion of the wave function is

$$\psi(\overrightarrow{x}',t) = \psi(\overrightarrow{x} + \triangle\overrightarrow{x}',t) = \psi(\overrightarrow{x},t) + \sum_{k=1}^{\infty}\frac{1}{k!}\nabla_d^k\psi(\overrightarrow{x},t) \tag{21}$$

where $k!$ is the factorial of k, $\triangle\overrightarrow{x}' = \overrightarrow{x}' - \overrightarrow{x}$, and

$$\nabla_d = \left(\triangle x_1'\frac{\partial}{\partial x_1} + \cdots + \triangle x_d'\frac{\partial}{\partial x_d}\right), \tag{22}$$

where $\triangle x_l' = x_l' - x_l$, for $l = 1, \cdots, d$, is the l-th component of $\triangle\overrightarrow{x}'$. As usual, in Eq. (21)

$$\psi(\overrightarrow{x}',t) = \psi(x_1',\cdots,x_d',t),$$

$$\psi(\overrightarrow{x} + \triangle\overrightarrow{x}', t) = \psi(x_1 + \triangle x_1', \cdots, x_d + \triangle x_d', t). \tag{23}$$

With the above extension to a d-dimensional lattice it is possible to rewrite all the one-dimensional equations. For example,

$$\frac{d\psi_{i_1\cdots i_d}(t)}{dt} = -\frac{\imath}{\hbar}\sum_{i_1'}\cdots\sum_{i_d'} H_{i_1\cdots i_d, i_1'\cdots i_d'}(t)\psi_{i_1'\cdots i_d'}(t), \quad \forall i_1,\cdots,i_d,i_1',\cdots,i_d' \in \Lambda_d, \tag{24}$$

where $i_1,\cdots,i_d,i_1',\cdots,i_d'$ are the components of the sites i, i' in a Λ_d lattice. Using a finite d-dimensional centered differences, extension of the discrete one-dimensional case can be obtained. The explicit form will be not considered here and only the continuum version will be analyzed. To this end, the definitions corresponding to the d-dimensional extensions are

$$\begin{aligned}\frac{\partial\psi(\overrightarrow{x},t)}{\partial t} &= -\frac{\imath}{\hbar}\int H(\overrightarrow{x},\overrightarrow{x}',t)\psi(\overrightarrow{x}',t)dV' \\ &= -\frac{\imath}{\hbar}\int H_1(\overrightarrow{x},\overrightarrow{x}')\psi(\overrightarrow{x}',t)dV' \\ &\quad -\frac{\imath}{\hbar}I_0H_0(\overrightarrow{x},\overrightarrow{x})\psi(\overrightarrow{x},t), \\ &\qquad \forall x_1,\cdots,x_d,x_1',\cdots,x_d' \in \Re,\end{aligned} \tag{25}$$

where, as in the one-dimensional case, $H(\overrightarrow{x},\overrightarrow{x}',t) = H_1(\overrightarrow{x},\overrightarrow{x}') + H_0(\overrightarrow{x},\overrightarrow{x})\frac{\psi(\overrightarrow{x},t)}{\psi(\overrightarrow{x}',t)}$, with $H_1(\overrightarrow{x},\overrightarrow{x}')$ and $H_0(\overrightarrow{x},\overrightarrow{x}')$ are both time independent, and

$$I_0 = \int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'} d\left(\triangle x_1'\right)\cdots d\left(\triangle x_d'\right)) = \left(2L_1'\right)\cdots\left(2L_d'\right).$$

Using the extension of the change of variables given in Eq. (15), it is easy to obtain

$$\begin{aligned}d(\triangle V') &= d(x_1'-x_1)\cdots d(x_d'-x_d) \\ &= d\left(\triangle x_l'\right)\cdots d\left(\triangle x_d'\right) \\ &= \frac{d(x_1'-x_1)}{dx_1'}dx_1'\cdots\frac{d(x_d'-x_d)}{dx_d'}dx_d' \\ &= dx_1'\cdots dx_d' \\ &= dV'.\end{aligned} \tag{26}$$

The final d-dimensional evolution equation can be written in the following convenient form

$$\frac{\partial\psi(\overrightarrow{x},t)}{\partial t} = -\frac{\imath}{\hbar}\int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'} H_1(\overrightarrow{x},\overrightarrow{x}')\psi(\overrightarrow{x}',t)\mathcal{D}\triangle' + H_{00} \quad \forall x_1,\cdots,x_d,x_1',\cdots,x_d' \in \Re, \tag{27}$$

where $H_{00} = -\frac{\imath}{\hbar}I_0H_0(\overrightarrow{x},\overrightarrow{x})\psi(\overrightarrow{x},t)$. It was also used $d(\triangle V') = d\left(\triangle x_l'\right)\cdots d\left(\triangle x_d'\right) = \mathcal{D}\triangle'$, in the second equality, which is very convenient when the expansion of $\psi(\overrightarrow{x}+\triangle\overrightarrow{x}',t)$ and $H_1(\overrightarrow{x},\overrightarrow{x}+\triangle\overrightarrow{x}')$, in a Taylor series, provides a set of elementary integrations as it was done in Eq. (18) for a one-dimensional lattice. Finally, using the Taylor series expansion given in Eq. (21) up to $k = 2$, and letting $H_1(\overrightarrow{x},\overrightarrow{x}+\triangle\overrightarrow{x}') = H_1(\overrightarrow{x},\overrightarrow{x}) + O(\triangle\overrightarrow{x}')$, a d-dimensional Schrödinger-like equation is obtained. As in the one-dimensional case, if only three term are kept, it is easy to find

$$\begin{aligned}\frac{\partial\psi(\overrightarrow{x},t)}{\partial t} &= H_{00} + \int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'} H_{11}\left(1 + \nabla_d + \frac{\nabla_d^2}{2}\right) \\ &\quad \times\psi(\overrightarrow{x},t)\mathcal{D}\triangle' \\ &= H_{00} + H_{11}I_0\psi(\overrightarrow{x},t) + I_1 + I_2, \\ &\qquad \forall x_1,\cdots,x_d,x_1',\cdots,x_d' \in \Re,\end{aligned} \tag{28}$$

where $H_{11} = -\frac{\imath}{\hbar}H_1(\overrightarrow{x},\overrightarrow{x})$,

$$\begin{aligned}I_1 &= -\frac{\imath}{\hbar}H_1(\overrightarrow{x},\overrightarrow{x})\int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'}\nabla\psi(\overrightarrow{x},t)\mathcal{D}\triangle' \\ &= H_{11}\int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'}\left(\triangle x_1'\frac{\partial}{\partial x_1} + \cdots + \triangle x_d'\frac{\partial}{\partial x_d}\right) \\ &\quad \times\psi(\overrightarrow{x},t)\mathcal{D}\triangle' \\ &= H_{11}\left(I_{1,1}\frac{\partial\psi(\overrightarrow{x},t)}{\partial x_1} + \cdots + I_{1,d}\frac{\partial\psi(\overrightarrow{x},t)}{\partial x_d}\right),\end{aligned} \tag{29}$$

and

$$\begin{aligned}I_2 &= -\frac{\imath}{\hbar}H_1(\overrightarrow{x},\overrightarrow{x})\int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'}\frac{\nabla_d^2}{2}\psi(\overrightarrow{x},t)\mathcal{D}\triangle' \\ &= \frac{H_{11}}{2}\int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'}\left(\triangle x_1'\frac{\partial}{\partial x_1} + \cdots + \triangle x_d'\frac{\partial}{\partial x_d}\right)^2 \\ &\quad \times\psi(\overrightarrow{x},t)\mathcal{D}\triangle'.\end{aligned} \tag{30}$$

It is not difficult to see that all the integrals on the right-hand side of Eq. (29) are zero. Remembering that $\frac{\partial\psi(\overrightarrow{x},t)}{\partial x_l}$ = constant $\forall l = 1,\cdots,d$, I_1 is a sum of elementary integrals. A generic multiple integral corresponding to the l-th term is a product of integrals like

$$\begin{aligned}I_{1,l} &= \int_{-L_1'}^{L_1'}\cdots\int_{-L_d'}^{L_d'}\triangle x_l' d\left(\triangle x_1'\right)\cdots d\left(\triangle x_d'\right)) \\ &= I_1'\cdots I_l'\cdots I_d' = 0, \quad \forall l = 1,\cdots,d,\end{aligned} \tag{31}$$

where

$$\begin{aligned}I_1' &= \triangle x_l'\int_{-L_1'}^{L_1'} d\triangle x_1' = \triangle x_l'\left[\triangle x_1'\right]_{-L_1'}^{L_1'} = 2L_1'\triangle x_l', \\ &\vdots\end{aligned}$$

$$I'_l = \int_{-L'_l}^{L'_l} \triangle x'_l d\triangle x'_l = \left[\frac{(\triangle x'_l)^2}{2}\right]_{-L'_1}^{L'_1} = 0, \quad \forall l = 1, \cdots, d,$$
$$\vdots$$
$$I'_d = \triangle x'_l \int_{-L'_d}^{L'_d} d\triangle x'_d = \triangle x'_l \left[\triangle x'_d)\right]_{-L'_d}^{L'_d} = 2L'_d \triangle x'_l. \tag{32}$$

Due to the fact that always one of the factors, in each of the d terms in Eq. (29), is zero because $I'_l = 0$ $\forall l = 1, \cdots, d$, then $I_{1,l} = 0$ $\forall l = 1, \cdots, d$ and consequently also $I_1 = 0$. After expanding the right-hand side in Eq. (30)

$$\begin{aligned} I_2 &= \frac{H_{11}}{2} \sum_{l.l'} \left(\int_{-L'_1}^{L'_1} \cdots \int_{-L'_d}^{L'_d} \triangle x'_l \triangle x'_{l'} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l \partial x_{l'}} \mathcal{D}\triangle' \right), \\ &= \frac{H_{11}}{2} \sum_{l.l'} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l \partial x_{l'}} \left(\int_{-L'_1}^{L'_1} \cdots \int_{-L'_d}^{L'_d} \triangle x'_l \triangle x'_{l'} \mathcal{D}\triangle' \right), \\ &= \frac{H_{11}}{2} \sum_{l.l'} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l \partial x_{l'}} (I_{l,l'}), \end{aligned} \tag{33}$$

where

$$\begin{aligned} I_{l,l'} &= \int_{-L'_1}^{L'_1} \cdots \int_{-L'_d}^{L'_d} \triangle x'_l \triangle x'_{l'} d\left(\triangle x'_1\right) \cdots d\left(\triangle x'_d\right) \\ &= I'_1 \cdots I'_l \cdots I'_d = 0, \qquad \forall l' \neq l, \end{aligned} \tag{34}$$

because

$$\begin{aligned} I'_{l'} &= \int_{-L'_l}^{L'_l} \triangle x'_l \triangle x'_{l'} d\left(\triangle x'_l\right) = \triangle x'_{l'} \int_{-L'_l}^{L'_l} \triangle x'_l d\left(\triangle x'_l\right) \\ &= \triangle x'_{l'} \left[\frac{\triangle x'^2_l}{2}\right]_{-L'_1}^{L'_l} = 0, \qquad \forall l' \neq l. \end{aligned} \tag{35}$$

On the other hand

$$\begin{aligned} I'_{l'} &= \int_{-L'_l}^{L'_l} \triangle x'_l \triangle x'_{l'} d\left(\triangle x'_l\right) = \int_{-L'_l}^{L'_l} \triangle x'^2_l d\left(\triangle x'_l\right) \\ &= \left[\frac{\triangle x'^3_l}{3}\right]_{-L'_1}^{L'_l} = 2\frac{L'^3_l}{3}, \qquad \forall l' = l, \end{aligned} \tag{36}$$

consequently

$$\begin{aligned} I_{l,l} &= I'_{l'} \overbrace{\int_{-L'_1}^{L'_1} \cdots \int_{-L'_d}^{L'_d}}^{(d-1)\text{ times}} d\left(\triangle x'_1\right) \cdots d\left(\triangle x'_d\right) \\ &= 2\frac{L'^3_l}{3} \left(\prod_{k \neq l} 2L'_k \right), \end{aligned} \tag{37}$$

where $I_{l,l'} = I_{l,l}$ was used, indicating that only those terms where $l' = l$ remain. The other $d - 1$ integrals are all of the form

$$I_k = \int_{-L'_k}^{L'_k} d\left(\triangle x'_k\right) = 2L'_k, \qquad \forall k \neq l.$$

The final result is

$$\begin{aligned} I_2 &= \frac{H_{11}}{2} \sum_{l.l} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l \partial x_l} (I_{l,l}), \\ &= \frac{H_{11}}{2} \sum_{l} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l^2} (I_{l,l}). \end{aligned} \tag{38}$$

Assuming $L'_1 = \cdots = L'_d = L'$ then $I_{l,l} = 2\frac{L'^3}{3} (2L')^{d-1}$ $\forall l = 1, \cdots, d$ and consequently

$$\begin{aligned} I_2 &= \frac{H_{11}}{2} (I_{l,l}) \sum_{l} \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_l^2}, \\ &= \frac{H_{11}}{2} \left(2\frac{L'^3}{3}\right) (2L')^{d-1} \nabla^2 \psi(\vec{x}, t), \\ &= -\frac{\imath}{\hbar} H_1(\vec{x}, \vec{x}) \left(\frac{L'^3}{3}\right) (2L')^{d-1} \nabla^2 \psi(\vec{x}, t), \end{aligned} \tag{39}$$

where

$$\nabla^2 = \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_1^2} + \cdots + \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_d^2}.$$

It is not difficult to see that for $d = 1$ the one-dimensional results are recovered. In order to connect the above evolution equation with a continuum Schrödinger-like equation it is necessary to make

$$\begin{aligned} \left[H_0(\vec{x}, \vec{x}) + H_1(\vec{x}, \vec{x})\right] I_0 &= V(\vec{x}), \\ H_1(\vec{x}, \vec{x}) \left(\frac{L'^3}{3}\right) (2L')^{d-1} &= \left(\frac{-\hbar^2}{2m_{\text{eff}}}\right), \end{aligned} \tag{40}$$

where $I_0 = \left(2L'_1\right) \cdots \left(2L'_d\right) = (2L')^d$. With these results Eq. (27) becomes

$$\frac{\partial \psi(\vec{x}, t)}{\partial t} = -\frac{\imath}{\hbar} \left(-\frac{\hbar^2}{2m_{\text{eff}}} \nabla^2 \psi(\vec{x}, t) + V(\vec{x}) \psi(\vec{x}, t) \right),$$

which is the well-known deterministic Schrödinger equation.

As expected, for $d = 1$ the results of Eqs. (16) and (17) or Eqs. (19) and (20), depending of the limits of the integrals, are recovered.

5 Other uses and extensions

Now it will be considered other possibilities of the present approach.

5.1 Evolution equation like the heat equation

For example, choosing the right-hand side of Eq. (40) in other different form, it could be found another evolution equation like the heat equation. Letting

$$\begin{aligned}\left[H_0(\vec{x},\vec{x})+H_1(\vec{x},\vec{x})\right]I_0 &= \mu\left(\frac{-\hbar}{\imath}\right),\\ H_1(\vec{x},\vec{x})\left(\frac{L'^3}{3}\right)(2L')^{d-1} &= \alpha\left(\frac{-\hbar}{\imath}\right),\end{aligned} \quad (41)$$

where α and μ are real coefficients, solving for $H_0(\vec{x},\vec{x})$ and $H_1(\vec{x},\vec{x})$ and after replace both in Eq. (27), it is possible to find the following evolution equation

$$\frac{\partial\psi(\vec{x},t)}{\partial t} = \alpha\nabla^2\psi(\vec{x},t)+\mu\psi(\vec{x},t). \quad (42)$$

5.2 Inclusion of nonlinear terms

Inclusion of nonlinear terms are possible letting

$$\begin{aligned}H(\vec{x},\vec{x}',t) &= \left[H_0(\vec{x},\vec{x}',t)\frac{\psi(\vec{x},t)}{\psi(\vec{x}',t)}+H_1(\vec{x},\vec{x}',t)\right]\\ &= \left[H_0(\vec{x},\vec{x},t)\frac{\psi(\vec{x},t)}{\psi(\vec{x}',t)}+H_1(\vec{x},\vec{x},t)\right].\end{aligned} \quad (43)$$

Both $H_0(\vec{x},\vec{x},t)$ and $H_1(\vec{x},\vec{x},t)$ are now time dependent Hamiltonians and were obtained after taking the Taylor series expansion up to the first term like $H_0(\vec{x},\vec{x}',t) = H_0(\vec{x},\vec{x},t)+O(\triangle\vec{x}')$ and $H_1(\vec{x},\vec{x}',t) = H_1(\vec{x},\vec{x},t)+O(\triangle\vec{x}')$. With these modifications both sides in Eq. (40) become

$$\begin{aligned}\left[H_0(\vec{x},\vec{x},t)+H_1(\vec{x},\vec{x},t)\right]I_0 &= \kappa_0\psi(\vec{x},t)\psi^*(\vec{x},t)\\ &= \kappa_0|\psi(\vec{x},t)|^2,\\ H_1(\vec{x},\vec{x},t)\left(\frac{L'^3}{3}\right)(2L')^{d-1} &= \kappa_2,\end{aligned} \quad (44)$$

where κ_0 and κ_2 are constants. Then, solving for $H_0(\vec{x},\vec{x},t)$ and $H_1(\vec{x},\vec{x},t)$ and after replacing both in Eq. (27), it is possible to find the following nonlinear deterministic Schrödinger equation

$$\frac{\partial\psi(\vec{x},t)}{\partial t} = -\frac{\imath}{\hbar}\left(\kappa_2\nabla^2\psi(\vec{x},t)+\kappa_0|\psi(\vec{x},t)|^2\psi(\vec{x},t)\right). \quad (45)$$

Of course, lot of other possibilities can be obtained by changing the right-hand side of Eq. (40) or Eq. (45).

5.3 Extension to a system of many particles

The extension to a many particles system is straightforward. The coordinates, the wave function, and the Laplacian, must be changed to

$$\begin{aligned}\vec{x} &= \vec{x}_1,\cdots,\vec{x}_N,\\ \psi(\vec{x},t) &= \psi(\vec{x}_1,\cdots,\vec{x}_N,t),\\ V(\vec{x},t) &= V(\vec{x}_1,\cdots,\vec{x}_N,t),\\ \nabla^2 &= \sum_{k=1}^{N}\nabla_k^2,\end{aligned}$$

where N is the number of particles, $\vec{x}_k = x_{1,k},\cdots,x_{d,k}$ are the coordinates of the k-th particle, and $\nabla_k^2 = \frac{\partial^2}{\partial x_{1,k}^2}+\cdots+\frac{\partial^2}{\partial x_{d,k}^2}$, for $k = 1,\cdots,N$.

6 The Schrödinger equation as a limiting case of the Schrödinger-like equation

It is possible to find the Schrödinger equation as a particular case of the Schrödinger-like equation as follows. Beginning with Eq. (13) and taking into account all the remaining terms in the Taylor series expansion of $\psi(x',t)$ in Eq. (12), it is possible to prove that all the remaining terms are zero, for a particular value of the limits of the integral. The remaining terms are

$$\begin{aligned}R(x',t) &= H_1'(x,x)\int_{-L_1''}^{L_1''}\sum_{n=3}^{\infty}\frac{\partial^n\psi(x,t)}{\partial x^n}\frac{(\triangle x')^n}{n!}d(\triangle x').\\ &= \sum_{n=3}^{\infty}\frac{\partial^n\psi(x,t)}{\partial x^n}I_n(L_1''),\end{aligned} \quad (46)$$

where

$$\begin{aligned}I_n(L_1'') &= H_1'(x,x)\int_{-L_1''}^{L_1''}\frac{(\triangle x')^n}{n!}d(\triangle x')\\ &= \frac{2H_1'(x,x)}{n!}\frac{(L_1'')^{n+1}}{n+1}.\end{aligned} \quad (47)$$

and it was used L_1'' instead of L_1' in order to avoid misunderstandings, like it was made in Eq. (18), because here again it will be assumed that the Hamiltonian is different from zero in an interval, which will be chosen in order to make the remaining term zero.

Using $H_1'(x,x)$, given in Eq. (16), the final result is

$$I_n(L_1'') = \frac{2}{n!}\frac{C}{(L_1'')^3}\frac{(L_1'')^{n+1}}{n+1} = C_n(L_1'')^{n-2}, \quad (48)$$

where

$$C = \frac{-3\hbar^2}{2m_{\text{eff}}},$$
$$C_n = \frac{2C}{n!(n+1)}. \quad (49)$$

It is obvious that for $L''_1 \rightarrow 0$, $R(x', t) \rightarrow 0$ in Eq. (46), then the Schrödinger-like equation becomes the Schrödinger equation. The extension to the d−dimensional case, using again L''_1 instead of L'_1, is straightforward. Beginning with the remainder terms of Eq. (28)

$$R(\overrightarrow{x'}, t) = \int_{-L''_1}^{L''_1} \cdots \int_{-L''_d}^{L''_d} H_{11} \left(\sum_{n=3}^{\infty} \frac{\nabla_d^n}{n!} \right) \psi(\overrightarrow{x}, t) \mathcal{D}\triangle'$$
$$= H_{11} \left(\sum_{n=3}^{\infty} \partial^n I_n \right) = \sum_{n=3}^{\infty} \partial^n H_{11} I_n, \quad (50)$$

where

$$\partial^n = \frac{1}{n!} \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} \frac{\partial^n \psi(\overrightarrow{x}, t)}{\partial x_{k_1} \cdots \partial x_{k_n}}, \quad (51)$$

$$I_n = \int_{-L''_1}^{L''_1} \cdots \int_{-L''_d}^{L''_d} \triangle'_{x_{k_1}} \cdots \triangle'_{x_{k_n}} \mathcal{D}\triangle', \quad (52)$$

and

$$\mathcal{D}\triangle' = d\left(\triangle'_{x_{k_1}}\right) \cdots d\left(\triangle'_{x_{k_n}}\right). \quad (53)$$

Using similar arguments like the ones that lead to Eq. (37) the multiple integral, assuming that $L''_1 = \cdots = L''_d$, is

$$I_n = 2\frac{L_1''^{(n+1)}}{n+1} \left(2L''_1\right)^{(d-1)}, \quad (54)$$

and using the value of $H_1(\overrightarrow{x}, \overrightarrow{x})$, obtained from Eq. (40), it is easy to see that

$$R_n = H_{11} I_n$$
$$= C_d \left(\frac{L_1''^3}{3} \left(2L''_1\right)^{(d-1)} \right)^{-1} 2\frac{L_1''^{(n+1)}}{n+1} \left(2L''_1\right)^{(d-1)}$$
$$= 6C_d \frac{L_1''^{(n-2)}}{n+1}, \quad (55)$$

where C_d is a constant independent of L''_1. It is not difficult to see that for $L''_1 \rightarrow 0$, $R_n \rightarrow 0 \quad \forall n \geq 3$ and consequently $R(\overrightarrow{x'}, t) \rightarrow 0$. This result proves that the Schrödinger equations can be obtained from Schrödinger-like equations as a limiting case.

7 Conclusions

The present paper deals with the derivation of deterministic Schrödinger-like equations from stochastic evolution equations after an average over realization both for discrete as well as continuum equations. The deterministic version can be compared to some previous results, where the proposed equations in [11], here are rigorously obtained from first principles. Also the Schrödinger equation can be obtained as a particular case of Schrödinger-like equations using a limiting procedure, as it was done in Section 6. Many other evolution equations can be obtained by simply changing the right-hand side of Eq. (44) and in the most general form like

$$\left[H_0(\overrightarrow{x}, \overrightarrow{x}, t) + H_1(\overrightarrow{x}, \overrightarrow{x}, t)\right] I_0 = F_0(\overrightarrow{x}, t),$$
$$H_1(\overrightarrow{x}, \overrightarrow{x}, t) \left(\frac{L'^3}{3} \right) (2L')^{d-1} = F_2(\overrightarrow{x}, t), \quad (56)$$

generates the nonlinear evolution equation

$$\frac{\partial \psi(\overrightarrow{x}, t)}{\partial t} = -\frac{\imath}{\hbar} \left(F_2(\overrightarrow{x}, t) \nabla^2 \psi(\overrightarrow{x}, t) + F_0(\overrightarrow{x}, t) \psi(\overrightarrow{x}, t) \right),$$

where $F_2(\overrightarrow{x}, t)$ and $F_0(\overrightarrow{x}, t)$ are arbitrary complex functions allowing to find all previous results, and many others, as special cases (see e.g. [17] for other examples of nonlinearity). It is easy to see that the new equations, in Eq. (56), are in general a system of four linear equations allowing to find the real and imaginary part of $H_1(\overrightarrow{x}, t)$ and $H_0(\overrightarrow{x}, t)$. Of course, this is a pure formal extension and the meaning, in each case, must be discussed. Note that all the discrete and continuum Schrödinger-like equations were obtained essentially using three main steps. Firstly, the assumption that the stochastic Hamiltonians and the wave functions are statistically independent, secondly, the splitting of the Hamiltonian in two terms like

$$H(\overrightarrow{x}, \overrightarrow{x}', t) = H_0(\overrightarrow{x}, \overrightarrow{x}', t) \frac{\psi(\overrightarrow{x}, t)}{\psi(\overrightarrow{x}', t)} + H_1(\overrightarrow{x}, \overrightarrow{x}', t)$$

and, thirdly, the expansion in a series up to an appropriate given order of $\triangle x'_l = x'_l - x_l$ for $l = 1, \cdots, d$.

Lot of additional extensions can be worked as, for example, the evolution equation of other operators as shown in the Appendix.

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Appendix

Here, it will be sketched another possible approach which can be obtained using the Taylor series expansion given in the last two equations in the introduction. The integral, given on the right-hand side in Eq. (9), can be written as

$$\begin{aligned} I &= \int H(x, x+\triangle x', t)\psi(x+\triangle x', t) d\triangle x' \\ &= \int \left(\sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} \frac{\Delta x'^m}{m!} \right) \times \left(\sum_{n=0}^{\infty} \frac{\partial^n \psi(x,t)}{\partial x^n} \frac{\Delta x'^n}{n!} \right) d\triangle x' \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\partial^m H(x,x,t)}{\partial x^m} \frac{\partial^n \psi(x,t)}{\partial x^n} \right) \int \frac{\Delta x'^m}{m!} \frac{\Delta x'^n}{n!} d\triangle x' \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\partial^m H(x,x,t)}{\partial x^m} \frac{\partial^n \psi(x,t)}{\partial x^n} I_{mn} \right) \\ &= \sum_{n=0}^{\infty} \left(H_{mn} \frac{\partial^n \psi(x,t)}{\partial x^n} \right) \qquad \forall x, x' \in \Re, \end{aligned} \tag{57}$$

where $d\triangle x' = d(x'-x) = dx'$, and assuming that the limits of $\triangle x'$ are $\triangle_i = x'_i - x$ and $\triangle_f = x'_f - x$, the integral is trivial and becomes

$$\begin{aligned} I_{mn} &= \int_{\triangle_i}^{\triangle_f} \frac{\Delta x'^m}{m!} \frac{\Delta x'^n}{n!} d\triangle x' \\ &= \frac{1}{m!n!} \int_{\triangle_i}^{\triangle_f} \Delta x'^{(m+n)} d\triangle x' \\ &= \frac{1}{m!n!} \left[\frac{\Delta x'^{(m+n+1)}}{(m+n+1)} \right]_{x'_i - x}^{x'_f - x} \\ &= \frac{1}{m!n!} \frac{\left(x'_f - x\right)^{(m+n+1)} - \left(x'_i - x\right)^{(m+n+1)}}{(m+n+1)}. \end{aligned} \tag{58}$$

Note that the integral is $I_{mn} = I_{mn}(x)$ and depends on x. Also it was defined

$$H_{mn} = \sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} I_{mn}. \tag{59}$$

A particularly interesting case is the one where the coefficients H_{mn} are finite for $n \leq 2$ and 0 otherwise. In this case

$$I = H_{m0}\psi(x,t) + H_{m1} \frac{\partial^1 \psi(x,t)}{\partial x^1} + H_{m2} \frac{\partial^2 \psi(x,t)}{\partial x^2}. \tag{60}$$

The coefficients that provide the Schrödinger equation are

$$\begin{aligned} H_{m0} &= \sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} I_{m0} = V(x), \\ H_{m1} &= \sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} I_{m1} = 0, \\ H_{m2} &= \sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} I_{m2} = \frac{-\hbar^2}{2m_{\text{eff}}}, \\ &\vdots \\ H_{mn} &= \sum_{m=0}^{\infty} \frac{\partial^m H(x,x,t)}{\partial x^m} I_{mn} = 0, \quad \forall n \geq 3. \end{aligned} \tag{61}$$

The present approach allows to find, if it is possible, not only the Schrödinger equation but also $\frac{\partial^m H(x,x,t)}{\partial x^m}$ $\forall m$, after solving the set of linear equations given in Eq. (61). Due to the fact that $|I_{mn}|$ becomes smaller for growing values of m and n, approximate values can be obtained making $m = n = p$, for p as large as we want. In matrix form the set of linear equations given in Eq. (61), for the case of $m = n = 4$, can be written as

$$\mathbf{I}_{44}\mathbf{D}_{41} = \mathbf{C}_{41}, \tag{62}$$

where

$$\mathbf{I}_{44} = \begin{pmatrix} \frac{1}{0!0!}\left(\frac{\triangle_f^1-\triangle_i^1}{1}\right) & \frac{1}{1!0!}\left(\frac{\triangle_f^2-\triangle_i^2}{2}\right) & \frac{1}{2!0!}\left(\frac{\triangle_f^3-\triangle_i^3}{3}\right) & \frac{1}{3!0!}\left(\frac{\triangle_f^4-\triangle_i^4}{4}\right) & \cdots \\ \frac{1}{0!1!}\left(\frac{\triangle_f^2-\triangle_i^2}{2}\right) & \frac{1}{1!1!}\left(\frac{\triangle_f^3-\triangle_i^3}{3}\right) & \frac{1}{2!1!}\left(\frac{\triangle_f^4-\triangle_i^4}{4}\right) & \frac{1}{3!1!}\left(\frac{\triangle_f^5-\triangle_i^5}{5}\right) & \cdots \\ \frac{1}{0!2!}\left(\frac{\triangle_f^3-\triangle_i^3}{3}\right) & \frac{1}{1!2!}\left(\frac{\triangle_f^4-\triangle_i^4}{4}\right) & \frac{1}{2!2!}\left(\frac{\triangle_f^5-\triangle_i^5}{5}\right) & \frac{1}{3!2!}\left(\frac{\triangle_f^6-\triangle_i^6}{6}\right) & \cdots \\ \frac{1}{0!3!}\left(\frac{\triangle_f^4-\triangle_i^4}{4}\right) & \frac{1}{1!3!}\left(\frac{\triangle_f^5-\triangle_i^5}{5}\right) & \frac{1}{2!3!}\left(\frac{\triangle_f^6-\triangle_i^6}{6}\right) & \frac{1}{3!3!}\left(\frac{\triangle_f^7-\triangle_i^7}{7}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{63}$$

$$\mathbf{D}_{41} = \begin{pmatrix} \frac{\partial^0 H(x,x,t)}{\partial x^0} \\ \frac{\partial^1 H(x,x,t)}{\partial x^1} \\ \frac{\partial^2 H(x,x,t)}{\partial x^2} \\ \frac{\partial^3 H(x,x,t)}{\partial x^3} \\ . \\ . \end{pmatrix}, \qquad \mathbf{C}_{41} = \begin{pmatrix} V(x) \\ 0 \\ \frac{-\hbar^2}{2m_{\text{eff}}} \\ 0 \\ . \\ . \end{pmatrix}. \tag{64}$$

Also, in general, matrix $\mathbf{I}_{pp}$ can be split as a sum of two symmetric ones like

$$\mathbf{I}_{pp} = \mathbf{I}^f_{pp} - \mathbf{I}^i_{pp}, \tag{65}$$

where the first four elements look like

$$\mathbf{I}^i_{44} = \begin{pmatrix} \frac{1}{0!0!}\binom{\triangle_i^1}{1} & \frac{1}{1!0!}\binom{\triangle_i^2}{2} & \frac{1}{2!0!}\binom{\triangle_i^3}{3} & \frac{1}{3!0!}\binom{\triangle_i^4}{4} \cdots \\ \frac{1}{0!1!}\binom{\triangle_i^2}{2} & \frac{1}{1!1!}\binom{\triangle_i^3}{3} & \frac{1}{2!1!}\binom{\triangle_i^4}{4} & \frac{1}{3!1!}\binom{\triangle_i^5}{5} \cdots \\ \frac{1}{0!2!}\binom{\triangle_i^3}{3} & \frac{1}{1!2!}\binom{\triangle_i^4}{4} & \frac{1}{2!2!}\binom{\triangle_i^5}{5} & \frac{1}{3!2!}\binom{\triangle_i^6}{6} \cdots \\ \frac{1}{0!3!}\binom{\triangle_i^4}{4} & \frac{1}{1!3!}\binom{\triangle_i^5}{5} & \frac{1}{2!3!}\binom{\triangle_i^6}{6} & \frac{1}{3!3!}\binom{\triangle_i^7}{7} \cdots \\ . & . & . & . \\ . & . & . & . \end{pmatrix}, \tag{66}$$

$$\mathbf{I}^f_{44} = \begin{pmatrix} \frac{1}{0!0!}\binom{\triangle_f^1}{1} & \frac{1}{1!0!}\binom{\triangle_f^2}{2} & \frac{1}{2!0!}\binom{\triangle_f^3}{3} & \frac{1}{3!0!}\binom{\triangle_f^4}{4} \cdots \\ \frac{1}{0!1!}\binom{\triangle_f^2}{2} & \frac{1}{1!1!}\binom{\triangle_f^3}{3} & \frac{1}{2!1!}\binom{\triangle_f^4}{4} & \frac{1}{3!1!}\binom{\triangle_f^5}{5} \cdots \\ \frac{1}{0!2!}\binom{\triangle_f^3}{3} & \frac{1}{1!2!}\binom{\triangle_f^4}{4} & \frac{1}{2!2!}\binom{\triangle_f^5}{5} & \frac{1}{3!2!}\binom{\triangle_f^6}{6} \cdots \\ \frac{1}{0!3!}\binom{\triangle_f^4}{4} & \frac{1}{1!3!}\binom{\triangle_f^5}{5} & \frac{1}{2!3!}\binom{\triangle_f^6}{6} & \frac{1}{3!3!}\binom{\triangle_f^7}{7} \cdots \\ . & . & . & . \\ . & . & . & . \end{pmatrix}, \tag{67}$$

and both square matrices are symmetrical. Of course, other Schrödinger-like evolution equation can be obtained changing the right-hand side of Eq. (61).

A possible generalization of the present approach can be to obtain the Taylor series expansion of $O\left(x, x', x'', ..., x^{(k)}, t\right)$ for any operator O and any k.

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