Quantum Trajectories: Dirac, Moyal and Bohm

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We recall Dirac's early proposals to develop a description of quantum phenomena in terms of a non-commutative algebra in which he suggested a way to construct what he called *quantum trajectories*. Generalising these ideas, we show how they are related to weak values and explore their use in the experimental construction of quantum trajectories. We discuss covering spaces which play an essential role in accounting for the *wave* properties of quantum particles. We briefly point out how new mathematical techniques take us beyond Hilbert space and into a deeper structure which connects with the algebras originally introduced by Born, Heisenberg and Jordan. This enables us to bring out the geometric aspects of quantum phenomena.
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1 Introduction

In a classic paper, Dirac [1] has drawn attention to the similarity of the *form* of the classical dynamical equations expressed in terms of commuting functions and the *form* of the corresponding non-commutative operator equations appearing in the quantum domain. The latter, essentially Heisenberg mechanics, can be represented by matrices and therefore form part of a non-commutative algebraic structure. This is in contrast to the Schrödinger approach which is represented in a formal Hilbert space structure, and leads to more familiar mathematics based on differential operators acting on continuous wave functions, the non-commutativity being taken care of in the form of the differential operators. These techniques, being more familiar to physicists, quickly generated results and placed the Schrödinger picture in prime position. This has led to the conclusion that the quantum *particle* appears more wave-like than the particles of classical dynamics.

In spite of this, Dirac felt that the replacement of commuting functions by non-commuting variables pointed to a deeper connection between the algebraic approach and classical mechanics and suggested that this relationship should be examined more closely. In making this proposal he realised that techniques necessary for handling non-commuting mathematics were not readily available. Nevertheless Dirac made some tentative suggestions on how to construct quantum expectation values when general non-commuting variables were involved. With these techniques at hand, he attempted to generalise the notion of a contact transformation to the quantum situation. Dirac thereby provided a method of constructing what he called the "trajectories of a quantum particle" based on a non-commutative structure and without using wave functions explicitly.

However these attempts were soon superseded by a third approach, the path integral method, which was proposed by Feynman [2] after he had read Dirac's paper. With the success of this approach, the notion of an actual quantum trajectory was dropped, particularly as Mott [3] had shown how the wave equation could be used to explain the trajectories seen in particle detectors like cloud chambers. This, together with the uncertainty principle, discouraged any further consideration of particle trajectories in the quantum domain. Moreover since no operational meaning could be given to such a notion, further discussion ceased. Thus Dirac's idea of constructing quantum trajectories was abandoned and even forgotten.

In the meantime, the more general debate concerning the completeness of the quantum formalism, initiated by Einstein, Podolsky and Rosen [4], continued unabated, focusing on the possibility of adding *hidden variables* and thereby allowing for the possibility of trajectories. This was in spite of von Neumann's [5] claim to have proved that such variables could not be used to explain the statistical properties of quantum processes without contradicting experimental results. However in 1952 a paper by Bohm [6, 7] appeared claiming that by simply splitting the Schrödinger equation into its real and imaginary parts, a more detailed account of quantum phenomena could, in fact, be given based on particle trajectories, an idea that had been anticipated notably by Madelung [8] and de Broglie [9].

Unfortunately the phrase *hidden variables* was used in the title of Bohm's paper whereas he actually introduced no additional parameters at all into the formalism. He had merely interpreted the existing formalism in a novel way. He was proposing a new dynamics that did not use the hydrodynamic model of Madelung [8]. In fact he had simply shown that the real part of the Schrödinger equation, under polar decomposition of the wave function, was of a form that looked remarkably like the classical Hamilton–Jacobi equation provided certain relations valid in the classical domain could be extended into the quantum domain. This equation, which we call the quantum Hamilton–Jacobi equation, enabled the straightforward calculation of what appeared to be *trajectories* as was demonstrated by Philippidis *et al.* [10] for an ensemble of particles constrained by certain experimental conditions such as defined in, for example, the two-slit experiment. Explanations of other quantum phenomena, again in terms of these *trajectories* followed, giving rise to an alternative understanding of these phenomena in a way that was thought to be impossible. (See Bohm and Hiley [11] and Holland [12].)

Thus contrary to expectation, these calculations demonstrated that it was possible to account for the interference phenomena in terms of collections of individual particle trajectories, although a deeper analysis raised the question of exactly what meaning could be given to the notion of a quantum particle following a trajectory. Unfortunately there seemed no way of experimentally determining these trajectories and so they remained a curiosity without experimental meaning. However some did embrace these ideas and developed a topic called *Bohmian mechanics* [13], using concepts that Bohm himself did not enthusiastically embrace, the latter arguing that something deeper was involved [14].

An examination of the two-slit experiment shows that the trajectories are not straight lines after they pass through the slits even though no classical potentials exist. The cause of these deviations could immediately be traced to the presence of the extra term appearing in the quantum Hamilton–Jacobi equation. At first, it was thought this extra term was merely an additional new classical potential since without it the particles would move in straight lines and no *fringes* would appear.

However a closer examination showed it to be very different from any known classical potential. It had no external point source; it was non-local, accounting for the effects of quantum entanglement and it reflected the properties of the immediate experimental arrangement, adding support to Bohr's notion of wholeness which he emphasised by demanding that the experimental conditions be included in the description. In many ways it seemed to be a new form of *inner energy* possessed by the particle, organising the flow lines in a novel way and suggesting a *formative* cause rather than the traditional efficient cause [11] (also see [15, 16]).

Unfortunately the inclusion of the phrase *hidden variables* in Bohm's paper, led to the belief that this was an attempt to return to a classical view of the world based on the old notion of mechanics, in contrast to the dominant view which was that such a return was impossible and a much more radical outlook was required. Bohm agreed and simply considered his proposal as a preliminary one providing a way to open up other, deeper possibilities.

However in the rather toxic atmosphere of the time, it was not realised that Bohm had added nothing new to *the mathematical structure* and was merely exploring the full implications of the quantum formalism in a different way. It should not be forgotten that the striking result of this approach was to bring out the notion of non-locality in entangled systems. Indeed it was Bohm's work that prompted Bell [17] to explore the wider consequences of this non-locality. Thus, far from returning to a classical picture, Bohm's work showed that the formalism contained many features that were clearly not classical and the whole approach was actually pointing to a radically new outlook.

Superficially, however, the Bohm approach did, at first glance, look naive as it seemed to provide no connection with the Heisenberg approach, not only in the sense that it seemed to violate the uncertainty principle, but it also seemed to avoid completely the non-commutative properties of the Heisenberg algebra. Rather than trying to understand how this approach produced results that were consistent with those deduced from the non-commuting operators, the discussion degenerated into a quasi-ideological battle between the two opposing views that emerged from *exactly the same mathematical structure*.

However a recent paper [18] pointed out that a non-commutative Heisenberg algebra had been further developed by von Neumann who showed how quantum phenomena emerged from a non-commutative phase space. This algebra was rediscovered by Moyal [19] who demonstrated that this approach could be understood as a generalisation of classical statistics to a new kind of statistical theory that was demanded by non-commutativity. Carried further, this non-commutativity seemed to require two time-dependent evolution equations [18]. In the Moyal algebra, for example, one of these is based on the Moyal bracket and the other on the Baker bracket [20]. In the classical limit, the first of these equations becomes the Liouville equation. While the second, based on the Baker bracket, reduces to the classical Hamilton–Jacobi equation. These two equations have an operator analogue based on the commutator and the anti-commutator, or Jordan product, which will be discussed in detail in section 3.5. When these equations are projected into the x-representation they become the quantum Liouville equation and the quantum Hamilton–Jacobi equation respectively. This immediately shows that the equations defining the Bohm approach are projections from a non-commutative space onto a shadow commutative phase space. (For a detailed discussion see Hiley [18].)

There is one further connection between the Moyal approach and the Bohm approach that is important to point out at this stage. The so-called guidance condition, $P_B = \nabla S$, also known as the Bohm momentum, which enables the direct calculation of the quantum trajectories, turns out to be the conditional momentum given by the Moyal joint distribution function $\hat{f}(\hat{X}, \hat{P})$. Here $(\hat{X}, \hat{P})$ are the operator equivalents of the coordinates of a cell in phase space, the so called *quantum blob* [21] although a deeper mathematical explanation exists [22], which we briefly introduce in section 3.2. Furthermore as we have already pointed out, one of the conditional time-development equations is identical to the quantum Hamilton–Jacobi equation, becoming the classical Hamilton–Jacobi equation in the appropriate limit. Thus the von Neumann–Moyal approach, based on a non-commutative algebra and the Bohm model are much more closely related than generally realised. In fact it could be argued that the Bohm approach forms an integral part of Heisenberg's matrix mechanics providing an intuitive account of the approach.

This brings us full circle to a classic paper by Dirac [1] which calls for a further investigation of the non-commutative Heisenberg approach. As we have already indicated, Dirac constructed a general distribution function for n non-commuting variables, which for the special case of two variables reduces to the Moyal distribution referred to above. Unfortunately Dirac incorrectly thought that Moyal's theory only dealt with operators of the form $e^{\iota(a\hat{X}+b\hat{P})}$ whereas, in fact, this term was used to define a distribution in phase space from which expectation values of *any function* of $(\hat{X}, \hat{P})$ can be calculated. This distribution is actually the Wigner function.

As has already been pointed out by one of us [23], the cross-Wigner function can be identified with the weak value of the momentum operator. In fact Dirac himself had implicitly introduced a form of weak value although he did not give it that name and saw his work as an opportunity to "discuss trajectories for the motion of a particle in quantum mechanics and thus make quantum mechanics more closely resemble classical mechanics"—his words, not ours [1].

2 Dirac's Quantum Trajectories

Regarding $\langle x_{t_f}|x_{t_0}\rangle$ as the probability amplitude of a particle travelling from position x_{t_0} to position x_{t_f} and travelling through a set of intermediate points, we can write

$$\langle x_{t_f}|x_{t_0}\rangle = \int \cdots \int \langle x_{t_f}|x_n\rangle dx_n \langle x_n|x_{n-1}\rangle dx_{n-1} \ldots dx_2 \langle x_2|x_1\rangle dx_1 \langle x_1|x_{t_0}\rangle. \tag{1}$$

where $\langle x_{i+1}|x_i\rangle$ is the propagator of the particle being at x_i at t_i and arriving at x_{i+1} at time t_{i+1}. Today we would

write this as

$$\langle x_{i+1}|x_i\rangle = \langle x_{i+1}|\exp[-\iota\hat{H}(t_{i+1} - t_i)]|x_i\rangle \tag{2}$$

but we will continue with the abridged notation for simplicity.

Thus a path is built up from a series of transitions between pairs of neighbouring points, (x_i, x_{i+1}) and the expectation value of an operator during each transition is given by

$$\langle x_{i+1}|\hat{F}(\hat{X}_i, \hat{X}_{i+1})|x_i\rangle = f(x_{i+1}, x_i)\langle x_{i+1}|x_i\rangle. \tag{3}$$

Furthermore we will assume the time $\epsilon = (t_{i+1} - t_i)$ to be small so that the trajectory can be divided into infinitesimal segments. Clearly we can now regard the element $\langle x_{i+1}|x_i\rangle_\epsilon$ as a propagator, which is written in the form

$$\langle x_{i+1}|x_i\rangle_\epsilon = \exp(\iota S_\epsilon(x_i, x_{i+1})/\hbar). \tag{4}$$

We will not, at this stage, identify the propagators with the Feynman propagators although clearly they are related. We will regard $S_\epsilon(x_i, x_{i+1})$ as a function generating the motion. Then, taking the momentum as an example, we find

$$\begin{aligned}
\langle x_{i+1}|\hat{P}_{i+1}|x_i\rangle_\epsilon &= -\iota\hbar\frac{\partial}{\partial x_{i+1}}\langle x_{i+1}|x_i\rangle_\epsilon \\
&= \frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_{i+1}}\langle x_{i+1}|x_i\rangle_\epsilon \\
&= \left\langle x_{i+1}\left|\frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_{i+1}}\right|x_i\right\rangle_\epsilon. \tag{5}
\end{aligned}$$

Thus from equation (3) we find

$$p_{i+1} = \frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_{i+1}}. \tag{6}$$

Similarly we can consider

$$\begin{aligned}
\langle x_{i+1}|\hat{P}_i|x_i\rangle_\epsilon &= \iota\hbar\frac{\partial}{\partial x_i}\langle x_{i+1}|x_i\rangle_\epsilon \\
&= -\frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_i}\langle x_{i+1}|x_i\rangle_\epsilon \\
&= -\left\langle x_{i+1}\left|\frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_i}\right|x_i\right\rangle_\epsilon \tag{7}
\end{aligned}$$

so that

$$p_i = -\frac{\partial S_\epsilon(x_i, x_{i+1})}{\partial x_i}. \tag{8}$$

Dirac suggested that p_i could be regarded as the momentum at the initial point (x_i, t_i) of the interval while p_{i+1} is the momentum at the final point (x_{i+1}, t_{i+1}), but clearly these are not eigenvalues of the momentum operators, so what are they?

2.1 The Classical Hamilton–Jacobi Theory

Let us proceed cautiously, first by recalling that the formulae (6) and (8) are reminiscent of classical Hamilton–Jacobi theory. In this theory, the function $S(x, x_0; t, t_0)$ generates a flow through a canonical transformation, or more technically, a symplectomorphism, f_{t,t_0}, such that

$$(x, p) \longmapsto f_{t,t_0}(x_0, p_0). \tag{9}$$

The word *symplectomorphism* is chosen to emphasise the basic underlying symplectic group. Specifically symplectomorphisms are elements of Ham($2n$), the group of Hamiltonian symplectomorphisms [24]. The flow is just another way to write Hamilton's equations of motion

$$\frac{d}{dt}f_t(x, p) = X_H(x, p) \tag{10}$$

where X_H is the Hamiltonian vector field

$$X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\right). \tag{11}$$

The time dependent flow is then defined as

$$f_{t,t_0}(x_0, p_0, t_0) = f_{t-t_0}(x_0, p_0, t_0) \tag{12}$$

so that equation (9) holds and defines functions $t \to x(t)$ and $t \to p(t)$ satisfying

$$\begin{aligned}
\dot{x}(t) &= \frac{\partial H(x(t), p(t), t)}{\partial p}; \tag{13} \\
\dot{p}(t) &= -\frac{\partial H(x(t), p(t), t)}{\partial x}. \tag{14}
\end{aligned}$$

The corresponding Hamilton–Jacobi equation is defined as

$$\frac{\partial S(x, x_0)}{\partial t} + H\left(x, \frac{\partial S(x, x_0)}{\partial x}, t\right) = 0. \tag{15}$$

Then for a free symplectomorphism $(x, p) = f_{t,t_0}(x_0, p_0)$, the following relations must be satisfied

$$\begin{aligned}
p &= \frac{\partial S(x, x_0; t, t_0)}{\partial x}; \\
p_0 &= -\frac{\partial S(x, x_0; t, t_0)}{\partial x_0}. \tag{16}
\end{aligned}$$

A remarkable similarity with quantum equations (6) and (8)? Yes, but notice that the generating function for f_{t,t_0} is $S(x, x_0; t, t_0)$ whereas the generating function for the quantum case uses the *exponential* of $S(x, x_0; t, t_0)$, namely, equation (4). This generates a different flow F_{t,t_0}, not in the group Ham($2n$) but in its *covering group*. In the linear case (namely, when the Hamiltonian is quadratic), f_{t,t_0} is an element of the symplectic group Sp($2n$), and F_{t,t_0} is an element of the metaplectic group Mp($2n$), the double cover of the symplectic group. What one can show is that there is a 1–1 correspondence between the continuous curves $t \to f_{t,t_0}$ in Sp($2n$) and the continuous curves $t \to F_{t,t_0}$ in Mp($2n$) [25].

The reference to a covering group is not totally unknown in physics. The notion of spin arises from a double cover, not of the symplectic group, but of the orthogonal group. In the case of spin, the spin group is just the double cover of the orthogonal group [26]. Similarly the metaplectic group provides a double cover for the symplectic group. Properties of both covering groups produce quantum effects that have been experimentally demonstrated [27–29]. So clearly the notion of a covering space plays a key role in quantum mechanics.

The relation between the symplectic group and its double cover is provided by the projection

$$\Pi : \text{Mp}(2n) \longrightarrow \text{Sp}(2n). \qquad (17)$$

Thus if f_{t,t_0} is the flow determined by the generating function $S(x, x_0; t, t_0)$ then, in the linear case

$$\psi(x, t) = F_{t,t_0} \psi(x_0, t_0) = A \int e^{iS_{t,t_0}(x, x_0; t, t_0)} \psi(x_0, t_0) d^3 x_0 \qquad (18)$$

where A is a convenient normalisation factor (this formula remains true for short times in the general case). One can show that $\psi(x, t)$ is a solution of the Schrödinger equation. For a complete account we need to extend the covering group to Ham($2n$) which is the non-linear generalisation of Sp($2n$). For a more detailed discussion see de Gosson and Hiley [25].

2.2 The Quantum Hamilton–Jacobi Equation

Having noticed the similarity between the Dirac equations (6) and (8) and the corresponding classical equations (16), let us now try to exploit this similarity in a different way. We start from equation (8), which we write in a slightly simpler notation as

$$p_0 = -\frac{\partial S_\epsilon}{\partial x_0}(x, x_0; t, t_0). \qquad (19)$$

We will now regard (x, x_0) as two independent variables. It follows from the implicit function theorem that equation (19) determines a function $x = x^\psi(t)$ provided

$$\frac{\partial^2 S_\epsilon}{\partial x \partial x_0} \neq 0.$$

We can then write

$$p_0 = -\frac{\partial S_\epsilon}{\partial x_0}(x^\psi(t), x_0, t, t_0), \qquad (20)$$

where x_0 and t_0 are to be viewed as independent parameters. Then let us define

$$p^\psi(t) = \frac{\partial S_\epsilon}{\partial x}(x^\psi(t), x_0, t, t_0). \qquad (21)$$

The functions $x^\psi(t)$ and $p^\psi(t)$ can then be shown to be solutions of the following Hamilton equations

$$\dot{x}^\psi(t) = \frac{\partial H^\psi}{\partial p}(x^\psi(t), p^\psi(t), t);$$

$$\dot{p}^\psi(t) = -\frac{\partial H^\psi}{\partial x}(x^\psi(t), p^\psi(t), t) \qquad (22)$$

with the initial conditions $x^\psi(t_0) = x_0$, $p^\psi(t_0) = p_0$; here we have written our Hamiltonian as H^ψ because it clearly cannot be the classical Hamiltonian as that would not have produced any quantum behaviour so what form will H^ψ take?

The corresponding Hamilton–Jacobi equation now becomes

$$\frac{\partial S}{\partial t} + H^\psi\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \qquad (23)$$

To show that this equation is equivalent to the pair of equations (22), first differentiate (23) with respect to x_0 and find

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\psi}{\partial x_0} = \frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\psi}{\partial p}\frac{\partial^2 S}{\partial x_0 \partial x} = 0 \qquad (24)$$

where we have used the chain rule.

Let us find the total differential of p_0 remembering we are regarding it as a parameter independent of time so that

$$\frac{dp_0}{dt} = \frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial^2 S}{\partial x \partial x_0}\dot{x} = 0. \qquad (25)$$

Subtracting (25) from (24), we get

$$\frac{\partial^2 S}{\partial x \partial x_0}\left(\frac{\partial H^\psi}{\partial p} - \dot{x}\right) = 0.$$

Since we have assumed that $\frac{\partial^2 S}{\partial x \partial x_0} \neq 0$ the first of Hamilton's equations emerges

$$\dot{x} = \frac{\partial H^\psi}{\partial p}. \qquad (26)$$

To obtain the second equation, we differentiate equation (23) with respect to x and find

$$\frac{\partial^2 S}{\partial x \partial t} + \frac{\partial H^\psi}{\partial x} + \frac{\partial H^\psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0. \qquad (27)$$

Introducing the canonical momentum

$$p(t) = \nabla_x S(x(t), x_0; t, t_0)$$

and differentiating with respect to t we obtain

$$\frac{\partial^2 S}{\partial x \partial t} = \dot{p} - \frac{\partial^2 S}{\partial x^2}\dot{x}.$$

Thus equation (27) can be written in the form

$$\dot{p}(t) - \frac{\partial^2 S}{\partial x^2}\dot{x} + \frac{\partial H^\psi}{\partial x} + \frac{\partial H^\psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0.$$

Taking into account Hamilton's first, we find Hamilton's second equation

$$\dot{p}(t) = -\frac{\partial H^\psi}{\partial x}. \qquad (28)$$

Hamilton's equations (26) and (28) will then give us an ensemble of trajectories from the equations (16) that Dirac assumed could be used to construct quantum trajectories.

The question therefore remains, "What is the form of H^ψ?" The earlier work had shown that if we consider the real part of the Schrödinger equation under polar decomposition of the wave function $\psi = R\exp[\iota S]$, we find the equation

$$\frac{\partial S(x,t)}{\partial t} + \frac{(\nabla S(x,t))^2}{2m} + Q^\psi(x,t) + V(x) = 0. \qquad (29)$$

This equation is identical in form to the classical Hamilton–Jacobi equation except that it contains an additional term, namely the quantum potential energy $Q^\psi(x,t)$. In other words this suggests that we identify

$$H^\psi = H + Q^\psi$$

where Q^ψ is given by

$$Q^\psi(x,t) = -\frac{1}{2m}\frac{\nabla^2 R(x,t)}{R(x,t)}. \qquad (30)$$

A more detailed discussion of this whole approach will be found in de Gosson [30].

Before going on to discuss in more detail the mathematical background to this approach and its relation to Dirac's proposals, we must make a point of clarification. Notice that the function $S_\epsilon(x, x')$ introduced in equation (4) is a two point function, namely a propagator, while the Bohm approach emerges from a one-point function, namely the wave function. This may not be a problem since the propagator $K(x, x'; t, t') = \psi(x, t)$ is the wave function, being simply the probability amplitude to get to (x, t) no matter what the initial point is [31]. Let us explore this relation in more detail.

2.3 Weak Values and Bohm Trajectories

Although the quantum Hamilton–Jacobi equation has been used to calculate trajectories [11, 12], their meaning has been controversial, and at times they have even been regarded as meaningless [32]. This is in spite of the fact that as the quantum potential becomes negligible the quantum trajectory deforms smoothly into a classical trajectory. There are two main factors contributing to the rejection of the notion of a quantum trajectory. Firstly there is the question of how we reconcile an uncertainty principle that arises from a fundamentally non-commutative structure. The need for such a revolutionary structure was made quite evident in the original work of Born, Dirac, Heisenberg and Jordan [33], yet equation (29) seems to imply that we need not concern ourselves with such complications. Unfortunately this is an illusion and although the approach does provide a useful, but partial insight into quantum phenomena, it is important to realise that we need to understand how this view is compatible with the underlying non-commutative structure. Secondly it has not previously been possible to investigate and construct these trajectories experimentally. With the appearance of weak values, this situation has now changed with the realisation that

- The weak value is not, in general, an eigenvalue of the operator under consideration.

- Weak values are complex numbers.

- The real part of the weak value of the momentum operator is identical to the momentum given in equation (6) where S_ϵ is identified with the phase of the wave function (the probability amplitude of getting to a point (x, t)).

- It is possible to measure weak values even though they are not eigenvalues, opening up the possibility of experimentally investigating the precise meaning of these trajectories.

The weak value of the momentum that is of interest in this paper can be written as

$$\frac{\langle x|\hat{P}|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = \nabla_x S(x,t) - \iota\nabla\rho(x,t)/2\rho(x,t) \qquad (31)$$

where we have chosen the polar decomposition of the wave function with $\rho(x,t) = |\psi|^2$. Notice that the real part of this weak value can be written as

$$\Re\langle\hat{P}\rangle_{w_{x_i,\psi}} = \Re\left[\frac{\langle x|\hat{P}|\psi\rangle}{\langle x|\psi\rangle}\right] = \frac{\partial S(x)}{\partial x} \qquad (32)$$

which suggests that there may be some connection with the p_{i+1} appearing in equation (6). Notice also that the

Dirac expressions emerge from a two-point propagator $S_\epsilon(x, x_0, t, t_0)$, not from the phase of a wave function. And what about equation (8) and the imaginary part of the weak value? Let us now look at these relations from another angle.

2.4 Relation of Weak Values to Non-Commutativity

Let us rewrite the expressions (6) and (8) in a different way to open up a new investigation

$$p_q = \frac{\langle q|\overrightarrow{P}|Q\rangle}{\langle q|Q\rangle} \quad \text{and} \quad p_Q = \frac{\langle Q|\overleftarrow{P}|q\rangle}{\langle Q|q\rangle}. \quad (33)$$

If we now form the sum $p_q + p_Q$, we find

$$[p_q + p_Q] = \left[\frac{\langle q|\overrightarrow{P}|Q\rangle}{\langle q|Q\rangle} + \frac{\langle Q|\overleftarrow{P}|q\rangle}{\langle Q|q\rangle}\right] \quad (34)$$

while the difference gives

$$[p_q - p_Q] = \left[\frac{\langle q|\overrightarrow{P}|Q\rangle}{\langle q|Q\rangle} - \frac{\langle Q|\overleftarrow{P}|q\rangle}{\langle Q|q\rangle}\right]. \quad (35)$$

If we change the notation $|q\rangle \rightarrow |x\rangle$ and $|Q\rangle \rightarrow |\psi\rangle$ we find

$$\frac{1}{2}\left[\frac{\langle x|\overrightarrow{P}|\psi\rangle}{\langle x|\psi\rangle} + \frac{\langle \psi|\overleftarrow{P}|x\rangle}{\langle \psi|x\rangle}\right] = \frac{\partial S(x, t)}{\partial x} = p_B(x, t) \quad (36)$$

while

$$\frac{1}{2}\left[\frac{\langle x|\overrightarrow{P}|\psi\rangle}{\langle x|\psi\rangle} - \frac{\langle \psi|\overleftarrow{P}|x\rangle}{\langle \psi|x\rangle}\right] = -\frac{\iota}{2\rho(x, t)}\frac{\partial \rho(x, t)}{\partial x} = -\iota p_o(x, t)$$

$$(37)$$

where we have written $\psi(x, t) = \sqrt{\rho(x, t)}\exp[\iota S(x, t)]$. Notice that both these momenta are real.

We may identify $p_B(x, t)$ with the Bohm or local momentum, while $p_o(x, t)$ can be identified with what Nelson [34] calls the *osmotic momentum*. The origin of the term *osmotic* has its roots in Nelson's attempts to derive the Schrödinger equation by considering a quantum particle undergoing a diffusive Brownian-type motion. Since a continuous derivative is ruled out in a stochastic motion, we have to distinguish between a forward derivative and a backward derivative.

In a non-commutative structure, we must distinguish between a left and a right translation, so that both momenta, (36) and (37), arise by combinations of the left and right translations of the momentum operator. This implies that the real and imaginary parts of a weak value result from the fact that we have, at the fundamental level, a non-commutative structure and by forcing this into a complex structure we have hidden some aspects of the deeper structure.

2.5 Some preliminary comments on the experimental situation

In a way we could claim that Dirac had essentially anticipated weak values, a fact that has already been pointed out by Salek, Schubert and Wiesner [35]. It should be noted that the weak value of the momentum is identical to the local momentum [36], a notion that has a long history going back to Landau [37] and London [38] in the early discussions of the superfluid properties of liquid helium. Because the local momentum could not be represented by a linear operator, London concluded that it was not a legitimate quantum observable as its value could not be measured in the standard way.

However that all changed when Wiseman [39] argued that the local momentum, being a weak value, could be measured in a process that Aharonov, Albert and Vaidman [40] called a *weak measurement*. The ideas lying behind the weak measurement were considerably clarified by Duck, Stevenson and Sudarshan [41]. Not only was the principle of a weak value and its measurement found to be correct, but an actual experiment carried out by Kocsis *et al.* [42] demonstrated how the local momentum could be measured in the interference region of a two-slit set up using a very weak electromagnetic source produced by a quantum dot. By measuring the weak value of the transverse local momentum at various positions in the region of interference, they were able to construct momentum flow lines, which resembled the Bohm trajectories calculated by Philippidis *et al.* [10] and therefore the flow lines were interpreted as "photon trajectories" [42].

Unfortunately this identification is not as straightforward as it seems at first sight. The trajectories constructed by Philippidis *et al.* were based on the Schrödinger equation, whereas photons must be described by a quantised Maxwell field. Again what appears to be a straightforward generalisation of the notion of trajectories for atoms to those of photons is not possible for reasons pointed out by Bohm, Hiley and Kaloyerou [43, 44]. Nevertheless the experimental determination of weak values has been demonstrated and experiments are in progress to measure weak values using atoms which, if successful, will open up a new debate in this area [45]. Let us therefore return to a discussion of the deeper mathematical structure lying behind these investigations.

3 The Non-commutative Phase Space

3.1 Connection Between Commutative and Non-commutative Phase Space

Even a glance at equation (29) shows that when the quantum potential energy Q^ψ is negligible and S identified with the classical action, we recapture the classical Hamilton–Jacobi equation. In other words, we change a non-commutative structure into a commutative structure. In terms of the argument that it is the covering group that determines the behaviour of quantum phenomena, the action is the second term in the expansion of $\exp[\iota S]$ (4). For a detailed discussion of the relationship between the classical action and the phase of the wave function see de Gosson [30].

Can we see, in a simple geometric way, how the space and its cover are related in a manner that helps with the understanding of the problem we are facing here without going into technical details? For the purposes of this paper formality is secondary, as a formal discussion already exists elsewhere [46].

To this end let us start by considering two points in a configuration space. Here we will simply write the coordinates of a single point as (x, t). Let us introduce a characteristic operator $\rho = |\psi\rangle\langle\psi|$, which in our configuration space we write as

$$\rho(x', x, t) = \psi^*(x', t)\psi(x, t).$$

In p-space we write

$$\phi(p, t) = \frac{1}{2\pi} \int \psi(x, t)e^{-\iota px}dx$$

so that

$$\rho(x', x, t) = \iint \phi^*(p', t)e^{-\iota p'x'}\phi(p, t)e^{\iota xp}dpdp'.$$

Let us now change coordinates and use

$$
\begin{aligned}
X &= \frac{x' + x}{2}, \\
\eta &= x' - x, \\
P &= \frac{p' + p}{2}, \\
\xi &= p' - p.
\end{aligned}
$$

Then

$$\rho(X, \eta, t) = \frac{1}{2\pi} \int \int \phi^*(P - \frac{\xi}{2}, t)\phi(P + \frac{\xi}{2}, t)e^{\iota X\xi}e^{\iota\eta P}d\xi dP$$

which we can write as

$$\rho(X, \eta, t) = \frac{1}{2\pi} \int F(X, P, t)e^{\iota\eta P}dP.$$

Taking the inverse Fourier transform of $\rho(X, \eta, t)$ will then provide us with a characteristic function of a process now unfolding in a phase space, where (X, P) are the coordinates, not of a particle, but of a region in configuration space characterised by a mean coordinate, X, and difference coordinate, η, and a mean momentum P and a difference ξ. These parameters provide a measure, in some limited sense, of the size of the region to which the energy of movement that is called *a particle* is confined.

3.2 Tangent Groupoids

It might seem that the introduction of a pair of points in configuration space is arbitrary. However a deeper analysis of the underlying non-commutative structure and its relation to the emergence of classical phase space has helped to clarify the geometric structure underlying quantum phenomena. Recently developed mathematical techniques (Connes [47] and Landsman [46]) based on asymptotic morphisms between C^*-algebras show the deep relations between the Moyal algebra, an algebra of a non-commutative phase space, and the Poisson algebra of classical phase space. One of the key ingredients of this approach is the tangent groupoid, a technique unfamiliar to the physics community so we will discuss this approach in a subsequent paper. We introduce these ideas here merely to indicate that there is a much richer structure underlying quantum phenomena that is just beginning to be revealed with the exploration of weak values. For preliminary details see Lopreore and Wyatt [48], Goldfarb *et al*. [49], Jozsa [50], Hiley [51], Dressel and Jordan [52], Shomroni *et al*. [53] and de Gosson *et al*. [54].

3.3 Return to Dirac

We now consider the Dirac proposal of finding quantum trajectories. Notice first that the two points, (x, x'), chosen were conjugate points. The corresponding operators of the mean variables (X, P) then satisfy the commutator $[X, P] = 0$, namely, this pair of operators are commutative

and therefore can have simultaneous eigenvalues which means a trajectory based on those operators can be well defined.

To carry the comparison further we have to note that Dirac also includes a pair of times (t, t'), whereas we have one time. In section 3.5 we will show how to generalise this approach to consider pairs of space-time points. A more general and detailed discussion will be found in Hiley [55, 56].

Replacing the notion of a particle by a region of active energy may, at first sight seem quite bizarre, but remember we are faced with a non-commutative phase space and this must of necessity include novel features. One of these is that the ordinary inner product must be replaced by a more general, non-commutative product that is translation and symplectic equivariant, associative, and non-local. There already exists a product with these properties, namely, the well known Moyal star product [57] to which we have already referred. A more detailed discussion of the relationship between the Moyal structure and the algebraic approach can be found in Hiley [18].

A further consequence of this relationship follows by performing a Fourier transformation on the characteristic operator to show it can be written in the form

$$F(X, P, t) = \frac{1}{2\pi} \int \psi^*(X - \frac{\eta}{2}, t) e^{-\iota \eta P} \psi(X + \frac{\eta}{2}, t) d\eta.$$

This will be immediately recognised as the Wigner function, a density matrix introduced for a different problem than the one we are discussing here [58]. For us it is the propagator of the time evolution of the process. There is no necessity to regard this function as a *probability distribution* as is done in quasi-classical quantum mechanics. We regard this as providing a weighting function for each operator under consideration and therefore no problem arises when it takes on negative values.

3.4 Connection with the Orthogonal Clifford Algebra

In section 2.1 we pointed out that quantum phenomena could be accounted for by going to the covering group of the symplectic group. This brings out the close geometric relation between the classical and the quantum behaviour. As we have already remarked a similar situation arises in the more familiar case of spin. Here the spin group, SU(2), is the covering group of the rotation group, SO(3). To analyse this structure, we have to go to the Clifford algebra, which, in this case, is a non-commutative algebra. All physicists are familiar with the anti-commutative structure of the Pauli σ matrices and the Dirac γ matrices but their use as geometric entities is novel.

These matrices are merely the representations of the generators of the respective Clifford algebras. The advantage of using the Clifford algebra is that the properties of the covering group can be obtained from the algebra itself. Indeed the covering group is the Clifford group which appears as a group of inner-automorphisms of the algebra and it turns out that one can work completely from within the algebra, with no need to represent properties in an abstract Hilbert space so that the wave function can be dispensed with. The wave function is not essential and has merely been introduced as an algorithm for calculating the probable outcome of a given system.

3.5 Non-commutative Time Development Equations

In the context of a non-commutative algebra, it is important, once again, to remember that we must distinguish between left and right translations. If we use the ket $|\psi\rangle$ to specify the state of the system then only left translations are possible. Furthermore it does not capture the fact that the wave function is a special case of a propagator as Feynman suggests. Therefore it was proposed that to obtain a description that allows both left and right time translations on an equal basis, we need to make a generalisation of the density operator, $\rho_{\psi,\phi} = |\psi\rangle\langle\phi|$. This operator characterises the process under investigation, and can be used in the special case of $\rho_{\psi,\psi}$, characterising the so-called *state of the system*. This also has the advantage of allowing a straightforward generalisation to mixed states. We will only be concerned with pure states in this paper when $\rho^2 = \rho$.

We will now assume that the equation for the left time translation is

$$\iota \frac{\partial |\psi\rangle}{\partial t} = \overrightarrow{H} |\psi\rangle,$$

while the right time translation is governed by the equation

$$-\iota \frac{\partial \langle\psi|}{\partial t} = \langle\psi| \overleftarrow{H}.$$

We seem not to have gained anything new compared with the standard approach because, surely this is simply writing down the Schrödinger equation and its complex conjugate equation and therefore apparently adds no new information. However when we consider the Pauli and Dirac equations, the left and right translations do not have such a simple relationship [26].

To see what new information these two equations contain, let us first form

$$\iota \frac{\partial |\psi\rangle}{\partial t} \langle\psi| = (\overrightarrow{H} |\psi\rangle)\langle\psi| \quad \text{and} \quad -\iota |\psi\rangle \frac{\partial \langle\psi|}{\partial t} = |\psi\rangle(\langle\psi| \overleftarrow{H}).$$

If we now add and subtract these two equations we obtain the following two equations, the first being

$$\iota \frac{\partial \rho}{\partial t} = [H, \rho]_-$$ (38)

which is recognised as Heisenberg's equation for the time development of the density operator. In the classical limit it becomes the Liouville equation describing the conservation of probability. The second equation, arising from the difference between the two equations, gives

$$\iota \left(|\psi\rangle \overleftrightarrow{\frac{\partial}{\partial t}} \langle \psi| \right) = [H, \rho]_+$$ (39)

where

$$\left(|\psi\rangle \overleftrightarrow{\frac{\partial}{\partial t}} \langle \psi| \right) = \left(\frac{\partial |\psi\rangle}{\partial t} \right) \langle \psi| - |\psi\rangle \left(\frac{\partial \langle \psi|}{\partial t} \right).$$

It should be remarked in passing that these equations are quite general and have been used in the case of the Pauli and Dirac equations [26]. It should also be noticed that in the two equations, (38) and (39), the quantum potential does not appear. For the full generalisation the kets and bras must be replaced by appropriate elements of the minimal left and right ideals in their respective algebras but we will not discuss this approach further here. The details can be found in Hiley and Callaghan [26] where it is shown how these elements can be represented by matrices.

To link up with the Schrödinger equation in its usual form, we must treat $|\psi\rangle$ as an element in the algebra, which can be polar decomposed, $\hat{\Psi} = \hat{R} \exp[\iota \hat{S}]$, and then inserted into equation (39) to find

$$2\rho \frac{\partial \hat{S}}{\partial t} + [H, \rho]_+ = 0.$$

This is just an equation for the conservation of energy that was first introduced by Dahl [59]. However if these equations are projected into a representation $|a\rangle$, we find the equations

$$i \frac{\partial P(a)}{\partial t} + \langle [\rho, H]_- \rangle_a = 0$$

and

$$2P(a) \frac{\partial S}{\partial t} + \langle [\rho, H]_+ \rangle_a = 0.$$

If we choose the x-representation, we find

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S_x}{m} \right) = 0.$$

Here $P(x,t)$ is the probability of finding the particle at (x,t) and S_x is the phase of the wave function in the x-representation. The second equation becomes

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m} \left(\frac{\partial S_x}{\partial x} \right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x} \left(\frac{\partial^2 R_x}{\partial x^2} \right) = 0.$$

Here the quantum potential appears for the first time. Thus the quantum potential emerges only when the time development equation is projected into a specific representation, in this case, the x-representation. Notice also that, on polar decomposition of the wave function, the two equations, (38) and (39), produce separately the real and the imaginary parts of the Schrödinger equation as two real but coupled equations.

If we were to choose to project these equations into the p-representation, we would obtain a different quantum potential. In fact the energy conservation equation now becomes

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2} x_r^2 - \frac{K}{2R_p} \left(\frac{\partial^2 R_p}{\partial p^2} \right) = 0.$$

Notice here we use $x_r = -\nabla_p S_p$, rather than $p_x = \nabla_x S_x$. A more detailed discussion of the consequences of a quantum potential appearing in the p-representation can be found in Brown and Hiley [60].

In this context the appearance of two projections at first sight seems rather strange and, for some, certainly unwelcome. However it restores the $x - p$ symmetry, the perceived lack of which Heisenberg [61] originally used as a criticism against the Bohm approach, but at the same time it destroys the comfortable intuitive form of the Bohm approach as the quantum process unfolding in an *a priori* given space-time. This opens up more radical approaches of the type that Bohm was already aware and was actively investigating [62]. In this paper we will not go into the interpretation of these results. Those interested will find details in [63].

Before concluding there are several features of this approach that should be noted. The two time-development equations (38) and (39) do not contain the complex wave function but correspond, in fact, to the imaginary and real parts respectively of the Schrödinger equation. Secondly by replacing the bras and kets by what Dirac [64] calls standard bras and standard kets, it can be shown that all the elements are contained within the algebra itself. An external Hilbert space is not needed. It is important to note this because interpretations based solely on Hilbert space vectors miss the deeper mathematical structure which is in need of a radically new interpretation. Thirdly, this approach does not require retro-causation which is very much in fashion at the time of writing. Fourthly, the Bohm approach is deeply imbedded in the

quantum formalism and the search for potential disagreements with the results of experiments predicted by the standard approach is futile.

Acknowledgements

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