
The Phase Space Formulation of Time-Symmetric Quantum Mechanics

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Time-symmetric quantum mechanics can be described in the Weyl–Wigner–Moyal phase space formalism by using the properties of the cross-terms appearing in the Wigner distribution of a sum of states. These properties show the appearance of a strongly oscillating interference between the pre-selected and post-selected states. It is interesting to note that the knowledge of this interference term is sufficient to reconstruct both states.

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1 Introduction

Time-symmetric quantum mechanics is an alternative formulation of quantum mechanics exhibiting fascinating and unconventional features whose potentialities have not yet been fully exploited; see [1–5], or the book [6] by Aharonov and Rohrlich. The present paper is a first step towards a formulation of time-symmetric quantum mechanics in terms of phase space concepts such as the Wigner distribution, and the ambiguity transform (the latter is essentially a Fourier transform of the Wigner distribution and is very much used in radar theory). To the best of our knowledge there are very few papers dis-

cussing the phase space approach (which is well-known in conventional quantum mechanics) in the context of time-symmetric quantum mechanics; exceptions to this state of affairs are our previous works [7, 8], and Gray’s Conference Proceedings note [9]. The advantage of the phase space approach is that it allows to calculate weak values using the classical observable; a problem that then arises (and which we will study in a forthcoming paper) is that the correspondence between a classical observable a and its *quantization* \hat{A} is by no means obvious: while it is true that most physicists rely on the Weyl scheme, there might be other physically meaningful ways to quantize a classical observable; for instance in [10, 11] we are advocating the use of Born–Jordan quantization, which predates Weyl quantization.

We will also focus on the reconstruction problem, which can roughly be stated as follows: knowing the interference between the pre-selected and post-selected states, can we reconstruct these states? We will see that knowing the cross-Wigner distribution of the pre-selected and post-selected states, suffices to uniquely determine both states. While this result is at first sight surprising, it is well-known in time-frequency analysis [12, 13] that it is possible to reconstruct a signal from the knowledge of its short-time Fourier transform with arbitrary window; the latter is closely related to the cross-Wigner transform.

Parts of this work (in particular the reconstruction formula Eq. 53) have been announced without motivations and proofs in previous work [7, 8]. We also mention that

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Lobo and Ribeiro [14] discussed weak values in the quantum phase space using methods that are very different from the Weyl–Wigner–Moyal formalism employed here.

1.1 Notation

We will work with systems having n degrees of freedom. Position or momentum variables are denoted $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$, respectively. The corresponding phase space variable is (x, p) . The scalar product $p_1 x_1 + \dots + p_n x_n$ is denoted by px . When integrating we will use, where appropriate, the volume elements $d^n x = dx_1 \dots dx_n$, $d^n p = dp_1 \dots dp_n$. The unitary \hbar -Fourier transform of a square-integrable function $\Psi(x)$ is

$$\tilde{\Psi}(p) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{n}{2}} \int e^{-\frac{i}{\hbar}px} \Psi(x) d^n x. \quad (1)$$

We denote by $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ and $\hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$ the operators defined by $\hat{x}_j \Psi = x_j \Psi$, $\hat{p}_j \Psi = -i\hbar \partial_{x_j} \Psi$.

1.2 The notion of weak value

In time-symmetric quantum mechanics the state of a system is represented by a two-state vector $\langle \Phi | | \Psi \rangle$ where the state $\langle \Phi |$ evolves backwards from the future and the state $| \Psi \rangle$ evolves forwards from the past. To make things clear, assume that at a time t_i an observable \hat{A} is measured and a non-degenerate eigenvalue was found: $| \Psi(t_i) \rangle = | \hat{A} = \alpha \rangle$; similarly at a later time t_f a measurement of another observable \hat{B} yields $| \Phi(t_f) \rangle = | \hat{B} = \beta \rangle$. Such a two-time state $\langle \Phi | | \Psi \rangle$ can be created as follows [1, 15]: Alice prepares a state $| \Psi(t_i) \rangle$ at initial time t_i . She then sends the system to an observer, Bob, who may perform any measurement he wishes to. The system is returned to Alice, who then performs a strong measurement with the state $| \Phi(t_f) \rangle$ as one of the outcomes. Only if this outcome is obtained, does Bob keep the results of his measurement.

Let now t be some intermediate time: $t_i < t < t_f$. Following the time-symmetric approach to quantum mechanics at this intermediate time the system is described by the *two* wavefunctions

$$\Psi = U_i(t, t_i) \Psi(t_i) \quad , \quad \Phi = U_f(t, t_f) \Phi(t_f) \quad (2)$$

where $U_i(t, t') = e^{-i\hat{H}_i(t-t')/\hbar}$ and $U_f(t, t') = e^{-i\hat{H}_f(t-t')/\hbar}$ are the unitary operators governing the evolution of the state before and after time t . Consider now the superposition of the two states $| \Psi \rangle$ and $| \Phi \rangle$ (which we suppose normalized); the expectation value

$$\langle \hat{A} \rangle_{\Psi+\Phi} = \frac{\langle \Psi + \Phi | \hat{A} | \Psi + \Phi \rangle}{\langle \Psi + \Phi | \Psi + \Phi \rangle} \quad (3)$$

of the observable \hat{A} in this superposition is obtained using the equality

$$N \langle \hat{A} \rangle_{\Psi+\Phi} = \langle \hat{A} \rangle_{\Phi} + \langle \hat{A} \rangle_{\Psi} + 2 \operatorname{Re} \langle \Phi | \hat{A} | \Psi \rangle \quad (4)$$

where $N = \langle \Psi + \Phi | \Psi + \Phi \rangle$. By definition, if $\langle \Phi | \Psi \rangle \neq 0$, the complex number

$$\langle \hat{A} \rangle_{\Phi, \Psi} = \frac{\langle \Phi | \hat{A} | \Psi \rangle}{\langle \Phi | \Psi \rangle} \quad (5)$$

is the *weak value* of \hat{A} .

1.3 What we will do

In the discussion above we have been working directly in terms of the wavefunctions Ψ and Φ ; now, a different kind of state description which is very fruitful, particularly in quantum optics, is provided by the Wigner distribution [11, 16–21]

$$W_{\Psi}(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py} \Psi\left(x + \frac{1}{2}y\right) \Psi^*\left(x - \frac{1}{2}y\right) d^n y; \quad (6)$$

the latter is directly related to the mean value of the observable $\langle \hat{A} \rangle_{\Psi} = \langle \Psi | \hat{A} | \Psi \rangle$ by Moyal's formula [11, 17–19, 22]

$$\langle \hat{A} \rangle_{\Psi} = \iint a(x, p) W_{\Psi}(x, p) d^n p d^n x \quad (7)$$

where $a(x, p)$ is the classical observable whose Weyl quantization is given by the Weyl–Moyal formula

$$\hat{A} = \left(\frac{1}{2\pi\hbar}\right)^n \iint \hat{a}(x, p) e^{\frac{i}{\hbar}(x\hat{x}+p\hat{p})} d^n p d^n x. \quad (8)$$

Here, we use the terminology *classical observable* in a very broad sense; a can be any complex integrable function, or even a tempered distribution that is an element of $\mathcal{S}'(\mathbb{R}^{2n})$, dual of the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ of rapidly decreasing functions. A direct calculation shows that we have

$$W_{\Psi+\Phi} = W_{\Phi} + W_{\Psi} + 2 \operatorname{Re} W_{\Psi, \Phi} \quad (9)$$

where the cross-term $W_{\Psi, \Phi}$ is given by

$$W_{\Psi, \Phi}(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py} \Psi\left(x + \frac{1}{2}y\right) \Phi^*\left(x - \frac{1}{2}y\right) d^n y. \quad (10)$$

The appearance of the term $W_{\Psi, \Phi}$ shows the emergence at time t of a strong interference between the pre-selected and the post-selected states $| \Psi \rangle$ and $| \Phi \rangle$. It is called the cross-Wigner distribution of Ψ, Φ , see [17, 18, 23] and the references therein. We are going to exploit the properties of $W_{\Psi, \Phi}$ to give an alternative working definition of the weak value $\langle \hat{A} \rangle_{\Phi, \Psi}$, namely

$$\langle \hat{A} \rangle_{\Phi, \Psi} = \frac{1}{\langle \Phi | \Psi \rangle} \iint a(x, p) W_{\Psi, \Phi}(x, p) d^n p d^n x \quad (11)$$

(see Eq. 20); here $a(x, p)$ is the classical observable whose Weyl quantization is the operator \hat{A} . Eq. 11 is justified by an extension of the averaging formula (Eq. 7) to pairs of

states: see Eq. 19, well-known in harmonic analysis. This allows us to interpret the function

$$\rho_{\Phi, \Psi}(x, p) = \frac{W_{\Psi, \Phi}(x, p)}{\langle \Phi | \Psi \rangle} \quad (12)$$

as a *complex* probability distribution. We thereafter notice that the cross-Wigner distribution can itself be seen, for fixed (x, p) , as a weak value, namely that of Grossmann and Royer's parity operator $\hat{T}_{GR}(x, p)$:

$$W_{\Psi, \Phi}(x, p) = (\pi\hbar)^n \langle \hat{T}_{GR}(x, p) \rangle_{\Psi, \Phi} \langle \Phi | \Psi \rangle \quad (13)$$

(see Eq. 36). Using this approach we prove the following Theorem 2: if $W_{\Psi, \Phi}$ is known, we can reconstruct (up to an unessential phase factor) the wave function Ψ (and hence the state $|\Psi\rangle$) with the use of

$$\Psi(x) = \frac{2^n}{\langle \Phi | \Lambda \rangle} \iint W_{\Psi, \Phi}(y, p) \hat{T}_{GR}(y, p) \Lambda(x) d^n p d^n y \quad (14)$$

where Λ is an arbitrary square-integrable function such that $\langle \Phi | \Lambda \rangle \neq 0$.

2 Weak Values in the Wigner Picture

2.1 The cross-Wigner transform

The cross-Wigner distribution is defined for all square-integrable functions Ψ, Φ ; it satisfies the generalized marginal conditions

$$\int W_{\Psi, \Phi}(x, p) d^n p = \Psi(x) \Phi^*(x) \quad (15)$$

$$\int W_{\Psi, \Phi}(x, p) d^n x = \tilde{\Psi}(p) \tilde{\Phi}^*(p) \quad (16)$$

provided that Ψ and Φ are in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$; these formulas reduce to the usual marginal conditions for the Wigner distribution when $\Psi = \Phi$. While W_{Ψ} is always real (though not non-negative, unless Ψ is a Gaussian), $W_{\Psi, \Phi}$ is a complex function, and we have $W_{\Psi, \Phi}^* = W_{\Phi, \Psi}$. The cross-Wigner distribution is widely used in signal theory and time-frequency analysis [17, 23]; its Fourier transform is the cross-ambiguity function familiar from radar theory [17, 24, 25]. Zurek [26] has studied $W_{\Psi, \Phi}$ when $\Psi + \Phi$ is a Gaussian cat-like state, and has shown that it is accountable for sub-Planck structures in phase space due to interference.

We now make the following elementary, but important remark: multiplying both sides of Eq. 9 by the classical observable $a(x, p)$ and integrating with respect to the x, p variables, we get, using Moyal's formula (Eq. 7),

$$\begin{aligned} \|\Phi + \Psi\| \langle \hat{A} \rangle_{\Psi + \Phi} &= \langle \hat{A} \rangle_{\Phi} + \langle \hat{A} \rangle_{\Psi} \\ &+ 2 \iint a(x, p) \operatorname{Re} W_{\Psi, \Phi}(x, p) d^n p d^n x. \end{aligned} \quad (17)$$

Comparing with Eq. 4 we see that

$$\operatorname{Re} \langle \Phi | \hat{A} | \Psi \rangle = \iint a(x, p) \operatorname{Re} W_{\Psi, \Phi}(x, p) d^n p d^n x. \quad (18)$$

It turns out that in the mathematical theory of the Wigner distribution [17, 18] one shows that the equality above actually holds not only for the real parts, but also for the purely imaginary parts, hence we always have

$$\langle \Phi | \hat{A} | \Psi \rangle = \iint a(x, p) W_{\Psi, \Phi}(x, p) d^n p d^n x. \quad (19)$$

An immediate consequence of this equality is that we can express the weak value $\langle \hat{A} \rangle_{\Phi, \Psi}$ in terms of the cross-Wigner distribution and the classical observable $a(x, p)$ corresponding to \hat{A} in the *Weyl quantization scheme*

$$\langle \hat{A} \rangle_{\Phi, \Psi} = \frac{1}{\langle \Phi | \Psi \rangle} \iint a(x, p) W_{\Psi, \Phi}(x, p) d^n p d^n x. \quad (20)$$

We emphasize that one has to be excessively careful when using formulas of the type (Eq. 20) (as we will do several times in this work): the function a crucially depends on the quantization procedure which is used (here Weyl quantization); we will come back to this essential point later, but here is a simple example which shows that things can get wrong if this rule is not observed: let $\hat{H} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2)$ be the quantization of the normalized harmonic oscillator $H(x, p) = \frac{1}{2}(x^2 + p^2)$ (we assume $n = 1$). While it is true that

$$\langle \hat{H} \rangle_{\Phi, \Psi} = \frac{1}{\langle \Phi | \Psi \rangle} \iint H(x, p) W_{\Psi, \Phi}(x, p) dp dx \quad (21)$$

it is in contrast *not true* that

$$\langle \hat{H}^2 \rangle_{\Phi, \Psi} = \frac{1}{\langle \Phi | \Psi \rangle} \iint H(x, p)^2 W_{\Psi, \Phi}(x, p) dp dx. \quad (22)$$

Suppose for instance that $\Psi = \Phi$ is the ground state of the harmonic oscillator: $\hat{H}\Psi = \frac{1}{2}\hbar\Psi$. We have

$$\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 = 0;$$

however use of Eq. 22 yields the wrong result

$$\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 = \frac{1}{4}\hbar^2.$$

The error comes from the inobservance of the prescription above: \hat{H}^2 is not the Weyl quantization of $H(x, p)^2$, but that of $H(x, p)^2 - \frac{1}{4}\hbar^2$ as is easily seen using the McCoy [27] rule

$$\widehat{x^r p^s} = \frac{1}{2^s} \sum_{k=0}^s \binom{s}{k} \hat{p}^{s-k} \hat{x}^r \hat{p}^k \quad (23)$$

and Born's canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ (see Shewell [28] for a discussion of related examples).

2.2 A complex phase space distribution

Let us now set

$$\rho_{\Phi,\Psi}(x, p) = \frac{W_{\Psi,\Phi}(x, p)}{\langle \Phi | \Psi \rangle}; \quad (24)$$

using the marginal conditions given by Eqs. 15,16 we get

$$\int \rho_{\Phi,\Psi}(x, p) d^n p = \frac{\Phi^*(x)\Psi(x)}{\langle \Phi | \Psi \rangle} \quad (25)$$

$$\int \rho_{\Phi,\Psi}(x, p) d^n x = \frac{\tilde{\Phi}^*(p)\tilde{\Psi}(p)}{\langle \Phi | \Psi \rangle} \quad (26)$$

hence the function $\rho_{\Phi,\Psi}$ is a complex probability distribution

$$\int \rho_{\Phi,\Psi}(x, p) d^n p d^n x = 1. \quad (27)$$

The weak value is given in terms of $\rho_{\Phi,\Psi}$ by

$$\langle \hat{A} \rangle_{\Phi,\Psi} = \int a(x, p) \rho_{\Phi,\Psi}(x, p) d^n p d^n x \quad (28)$$

which reduces to Eq. 7 in the case of an ideal measurement, namely $\Phi = \Psi$. The practical meaning of these relations is the following [5]: the readings of the pointer of the measuring device will cluster around the value

$$\text{Re}\langle \hat{A} \rangle_{\Phi,\Psi} = \int \text{Re}(a(x, p) \rho_{\Phi,\Psi}(x, p)) d^n p d^n x \quad (29)$$

while the quantity

$$\text{Im}\langle \hat{A} \rangle_{\Phi,\Psi} = \int \text{Im}(a(x, p) \rho_{\Phi,\Psi}(x, p)) d^n p d^n x \quad (30)$$

measures the shift in the variable conjugate to the pointer variable. In an interesting paper [29] Feyereisen discusses some aspects of the complex distribution $\rho_{\Phi,\Psi}$.

2.3 The cross-Wigner transform as a weak value

Let $\hat{T}(x_0, p_0) = e^{-\frac{i}{\hbar}(p_0\hat{x} - x_0\hat{p})}$ be the Heisenberg operator; it is a unitary operator whose action on a wavefunction Ψ is given by

$$\hat{T}(x_0, p_0)\Psi(x) = e^{\frac{i}{\hbar}(p_0x - \frac{1}{2}p_0x_0)}\Psi(x - x_0). \quad (31)$$

It has the following simple dynamical interpretation [18, 21]: $\hat{T}(z_0)$ is the time-one propagator for the Schrödinger equation corresponding to the translation Hamiltonian $\hat{H}_0 = x_0\hat{p} - p_0\hat{x}$. An associated operator is the Grossmann–Royer reflection operator (or displacement parity operator) [18, 30, 31] given by

$$\hat{T}_{\text{GR}}(x_0, p_0) = \hat{T}(x_0, p_0)R^\vee\hat{T}(x_0, p_0)^\dagger \quad (32)$$

where R^\vee changes the parity of the function to which it is applied: $R^\vee\Psi(x) = \Psi(-x)$; the explicit action of $\hat{T}_{\text{GR}}(z_0)$ on wavefunctions is easily obtained using Eq. 31 and one finds

$$\hat{T}_{\text{GR}}(x_0, p_0)\Psi(x) = e^{\frac{2i}{\hbar}p_0(x-x_0)}\Psi(2x_0 - x). \quad (33)$$

Now, a straightforward calculation shows that the Wigner distribution W_Ψ is (up to an unessential factor), the expectation value of $\hat{T}_{\text{GR}}(x_0, p_0)$ in the state $|\Psi\rangle$; in fact (dropping the subscripts 0)

$$W_\Psi(x, p) = \left(\frac{1}{\pi\hbar}\right)^n \langle \hat{T}_{\text{GR}}(x, p)\Psi | \Psi \rangle. \quad (34)$$

More generally, a similar calculation shows that the cross-Wigner transform is given by

$$W_{\Psi,\Phi}(x, p) = \left(\frac{1}{\pi\hbar}\right)^n \langle \hat{T}_{\text{GR}}(x, p)\Phi | \Psi \rangle \quad (35)$$

and can hence be viewed as a transition amplitude. Taking Eq. 5 into account we thus have

$$W_{\Psi,\Phi}(x, p) = (\pi\hbar)^n \langle \hat{T}_{\text{GR}}(x, p) \rangle_{\Psi,\Phi} \langle \Phi | \Psi \rangle; \quad (36)$$

this relation immediately implies, using definition (24) of the complex probability distribution $\rho_{\Phi,\Psi}$, the important equality

$$\rho_{\Phi,\Psi}(x, p) = (\pi\hbar)^n \langle \hat{T}_{\text{GR}}(x, p) \rangle_{\Psi,\Phi} \quad (37)$$

which can in principle be used to determine $\rho_{\Phi,\Psi}$.

As already mentioned, the cross-ambiguity function $A_{\Psi,\Phi}$ is essentially the Fourier transform of $W_{\Psi,\Phi}$; in fact

$$A_{\Psi,\Phi} = \mathcal{F}_\sigma W_{\Psi,\Phi}, \quad W_{\Psi,\Phi} = \mathcal{F}_\sigma A_{\Psi,\Phi} \quad (38)$$

where \mathcal{F}_σ is the symplectic Fourier transform: if $a = a(x, p)$ then $\mathcal{F}_\sigma a(x, p) = \tilde{a}(p, -x)$ where \tilde{a} is the ordinary $2n$ -dimensional \hbar -Fourier transform of a ; explicitly

$$\mathcal{F}_\sigma a(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \iint e^{-\frac{i}{\hbar}(xp' - p'x)} a(x', p') d^n p' d^n x'. \quad (39)$$

Both equalities in Eq. 38 are equivalent because the symplectic Fourier transform is involutive, and hence its own inverse. While the cross-Wigner distribution is a measure of *interference*, the cross-ambiguity function is rather a measure of *correlation*. One shows [11, 17, 18, 23] that $A_{\Psi,\Phi}$ is explicitly given by

$$A_{\Psi,\Phi}(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\Psi\left(y + \frac{1}{2}x\right)\Phi^*\left(y - \frac{1}{2}x\right)d^n y. \quad (40)$$

The cross-ambiguity function is easily expressed using the Heisenberg operator instead of the Grossmann–Royer operator as

$$A_{\Psi,\Phi}(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \langle \hat{T}(x, p)\Phi | \Psi \rangle. \quad (41)$$

The following important result shows that the knowledge of the classical observable a allows us to determine the weak value of the corresponding Weyl operator using the weak value of the Grossmann–Royer (respectively the Heisenberg) operator:

Theorem 1. *Let \hat{A} be the Weyl quantization of the classical observable a . We have*

$$\langle \hat{A} \rangle_{\Phi, \Psi} = \left(\frac{1}{\pi \hbar} \right)^n \iint a(x, p) \langle \hat{T}_{GR}(x, p) \rangle_{\Phi, \Psi} d^n p d^n x \quad (42)$$

and

$$\langle \hat{A} \rangle_{\Phi, \Psi} = \left(\frac{1}{2\pi \hbar} \right)^n \iint \mathcal{F}_\sigma a(x, p) \langle \hat{T}(x, p) \rangle_{\Phi, \Psi} d^n p d^n x. \quad (43)$$

Proof. In view of Moyal’s formula (Eq. 19) we have

$$\langle \Phi | \hat{A} | \Psi \rangle = \iint a(x, p) W_{\Psi, \Phi}(x, p) d^n p d^n x \quad (44)$$

that is, taking Eq. 35 into account

$$\langle \Phi | \hat{A} | \Psi \rangle = \left(\frac{1}{\pi \hbar} \right)^n \iint a(x, p) \langle \hat{T}_{GR}(x, p) \Phi | \Psi \rangle d^n p d^n x \quad (45)$$

hence Eq. 42; Eq. 43 is obtained in a similar way, first applying the Plancherel formula to the right-hand side of Eq. 44, then applying the first identity given by Eq. 38, and finally using Eq. 41. \square

Notice that the formulas above immediately yield the well-known [11, 17, 18, 21] representations of the operator \hat{A} in terms of the Grossmann–Royer and Heisenberg operators:

$$\hat{A} = \left(\frac{1}{\pi \hbar} \right)^n \iint a(x, p) \hat{T}_{GR}(x, p) d^n p d^n x \quad (46)$$

and

$$\hat{A} = \left(\frac{1}{2\pi \hbar} \right)^n \iint \mathcal{F}_\sigma a(x, p) \hat{T}(x, p) d^n p d^n x. \quad (47)$$

3 The Reconstruction Problem

3.1 Lundeen’s experiment

In 2012, Lundeen and his co-workers [32] determined the wavefunction by weakly measuring the position, and thereafter performing a strong measurement of the momentum. They considered the following experiment on a particle: a weak measurement of x is performed which amounts to applying the projection operator $\hat{\Pi}_x = |x\rangle\langle x|$ to the pre-selected state $|\Psi\rangle$; thereafter they perform a strong measurement of momentum, which yields the value p_0 ,

that is $\Phi(x) = e^{\frac{i}{\hbar} p_0 x}$. The result of the weak measurement is thus

$$\langle \hat{\Pi}_x \rangle_{\Psi, \Phi} = \frac{\langle p_0 | x \rangle \langle x | \Psi \rangle}{\langle p_0 | \Psi \rangle} = \left(\frac{1}{2\pi \hbar} \right)^{\frac{n}{2}} \frac{e^{-\frac{i}{\hbar} p_0 x} \Psi(x)}{\tilde{\Psi}(p_0)} \quad (48)$$

where $\tilde{\Psi}$ the Fourier transform of Ψ . Since the value of p_0 is known we get

$$\Psi(x) = \frac{1}{k} e^{\frac{i}{\hbar} p_0 x} \langle \hat{\Pi}_x \rangle_{\Psi, \Phi} \quad (49)$$

where $k = (2\pi \hbar)^{\frac{n}{2}} \tilde{\Psi}(p_0)$; Eq. 49 thus allows to determine $\Psi(x)$ by scanning through the values of x . Thus, by reducing the disturbance induced by measuring the position and thereafter performing a sharp measurement of momentum we can reconstruct the wavefunction pointwise. In [33] Lundeen and Bamber generalize this construction to mixed states and arbitrary pairs of observables. Using the complex distribution $\rho_{\Psi, \Phi}(x, p)$ defined above it is easy to recover Eq. 49 of Lundeen *et al.* In fact, choose $a(x, p) = \Pi_{x_0}(x, p) = \delta(x - x_0)$; its Weyl quantization

$$\hat{\Pi}_{x_0} \Psi(x) = \Psi(x_0) \delta(x - x_0)$$

is the projection operator: $\hat{\Pi}_{x_0} |\Psi\rangle = \Psi(x_0) |x_0\rangle$. Using the elementary properties of the Dirac delta function together with the marginal property 25, Eq. 28 becomes

$$\begin{aligned} \langle \hat{\Pi}_{x_0} \rangle_{\Phi, \Psi} &= \int \delta(x - x_0) \rho_{\Phi, \Psi}(x, p) d^n p d^n x \\ &= \int \rho_{\Phi, \Psi}(x_0, p) d^n p \\ &= \frac{\Phi^*(x_0) \Psi(x_0)}{\langle \Phi | \Psi \rangle} \end{aligned}$$

which is Eq. 48 since $\Phi(x_0) = e^{\frac{i}{\hbar} p_0 x_0}$; Eq. 49 follows.

3.2 Reconstruction: the Weyl–Wigner–Moyal approach

It is well-known [17, 18] that the knowledge of the Wigner distribution W_Ψ uniquely determines the state $|\Psi\rangle$; this is easily seen by noting that W_Ψ is essentially a Fourier transform and applying the Fourier inversion formula, which yields

$$\Psi(x) \Psi^*(x') = \int e^{\frac{i}{\hbar} p(x-x')} W_\Psi \left[\frac{1}{2}(x+x'), p \right] d^n p; \quad (50)$$

one then chooses x' such that $\Psi(x') \neq 0$, which yields the value of $\Psi(x)$ for arbitrary x . The same procedure applies to the cross-Wigner transform (Eq. 10); one finds that

$$\Psi(x) \Phi^*(x') = \int e^{\frac{i}{\hbar} p(x-x')} W_{\Psi, \Phi} \left[\frac{1}{2}(x+x'), p \right] d^n p. \quad (51)$$

Notice that if we choose $x' = x$ we recover the generalized marginal condition (Eq. 15) satisfied by the cross-Wigner distribution.

Thus, the knowledge of $W_{\Psi,\Phi}$ and Φ is in principle sufficient to determine the wavefunction Ψ . Here is a stronger statement which shows that the state $|\Psi\rangle$ can be reconstructed from $W_{\Psi,\Phi}$ using an *arbitrary* auxiliary state $|\Lambda\rangle$ non-orthogonal to $|\Phi\rangle$:

Theorem 2. *Let Λ be an arbitrary vector in $L^2(\mathbb{R}^n)$ such that $\langle\Phi|\Lambda\rangle \neq 0$. We have*

$$\Psi(x)\langle\Phi|\Lambda\rangle = 2^n \iint e^{\frac{2i}{\hbar}p(x-y)} W_{\Psi,\Phi}(y,p)\Lambda(2y-x)d^n p d^n y \quad (52)$$

that is

$$\Psi(x) = \frac{2^n}{\langle\Phi|\Lambda\rangle} \iint W_{\Psi,\Phi}(y,p)\hat{T}_{GR}(y,p)\Lambda(x)d^n p d^n y; \quad (53)$$

equivalently,

$$\Psi(x) = 2^n \frac{\langle\Psi|\Phi\rangle}{\langle\Phi|\Lambda\rangle} \iint \rho_{\Psi,\Phi}(y,p)\hat{T}_{GR}(y,p)\Lambda(x)d^n p d^n y. \quad (54)$$

Proof. By a standard continuity and density argument it is sufficient to assume that Ψ, Φ, Λ are in $\mathcal{S}(\mathbb{R}^n)$. Using Eq. 51 we have

$$\Psi(x)\langle\Phi|\Lambda\rangle = \iint e^{\frac{i}{\hbar}p(x-x')} W_{\Psi,\Phi}(\frac{1}{2}(x+x'), p)\Lambda(x')d^n p d^n x'.$$

Setting $y = \frac{1}{2}(x + x')$ we get Eq. 52 and hence Eq. 53 in view of the explicit formula for the Grossmann–Royer parity operator (Eq. 33). \square

Here is an example: viewing $\langle\Phi|\Lambda\rangle$ as the distributional bracket $\langle\Lambda, \Phi^*\rangle$ we may choose $\Lambda(x) = \delta(x - x_0)$. This yields $\langle\Lambda, \Phi^*\rangle = \Phi^*(x_0)$ and the right-hand side of Eq. 52 is just the integral

$$\int e^{\frac{i}{\hbar}p(x-x')} W_{\Psi,\Phi}(\frac{1}{2}(x+x'), p)d^n p$$

hence we recover Eq. 51 as a particular case.

4 Discussion

We have been able to give a complete characterization of the notion of weak value in terms of the Wigner distribution, which is intimately related to the Weyl quantization scheme through Moyal’s formula (Eq. 7). There are however other possible physically meaningful quantization schemes; the most interesting is certainly that of Born–Jordan [34, 35] mentioned in the introduction; the latter

plays an increasingly important role in quantum mechanics and in time-frequency analysis [7, 8, 10, 11, 36–38], and each of these leads to a different phase space formalism, where the Wigner distribution has to be replaced by more general element of the *Cohen class* [39, 40]. Unexpected difficulties however arise, especially when one deals with the reconstruction problem; these difficulties have a purely mathematical origin, and are related to the division of distributions (for a mathematical analysis of the nature of these difficulties, see [38]). The reconstruction problem for general phase space distributions will be addressed in a forthcoming publication. It should also be mentioned that Hiley and Cohen have proposed in [41] an approach to retrodiction from the perspective of the Einstein–Podolsky–Rosen–Bohm experiment; it is possible that this approach could be studied from the point of view of the techniques developed here.

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