Is Bohm's Interpretation Consistent with Quantum Mechanics?

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The supposed equivalence of the conventional interpretation of quantum mechanics with Bohm's interpretation is generally demonstrated only in the coordinate representation. It is shown, however, that in the momentum representation this equivalence is not valid.
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1 Introduction

Recently, there has been a renewed interest in David Bohm's interpretation of *non-relativistic* quantum mechanics [1–4] and many pedagogical papers on this topic have appeared [5–17], while online, arXiv.org lists over 200 submissions on this topic during the past ten years. Bohm claimed that "as long as the present general form of Schrödinger's equation is retained the physical results obtained with [this] suggested alternative interpretation are precisely the same as those obtained with the usual interpretation", and that his interpretation "leads to precisely the same results for all physical processes as does the usual interpretation" [1, p.166]. Similar assertions also have been made in references [5–17], but this equivalence is usually demonstrated only in the coordinate representa-

tion, while the implications of Bohm's interpretation in the momentum representation are usually ignored. While there have been some criticisms in the past of Bohm's interpretation of quantum mechanics [18, 19], we give here an elementary proof that the momentum distribution in this interpretation differs from that in standard quantum mechanics. We show that the definition of particle velocity in this interpretation, implies that the product of mass times velocity is not equal to momentum, which is inconsistent with both classical and quantum mechanics. The word "consistent" is used here in accordance to its definition in the World English Dictionary: "A set of statements capable of all being true at the same time or under the same interpretation".

2 Bohmian mechanics differs from conventional quantum mechanics in the momentum representation

In Bohm's interpretation of quantum mechanics, the velocity of a particle with mass m is given by

$$\vec{v}_B = \vec{\nabla} S / m \tag{1}$$

where $S/\hbar$ is the phase of the wave function ψ obtained by solving the time dependent Schrödinger equation. According to Bohm,

$$\vec{v}_B = \frac{d\vec{q}}{dt}, \tag{2}$$

where $\vec{q}$ is the time dependent coordinate for the position of the particle, and Equation 1 becomes a first order differential equation that determines $\vec{q}$ as a function of time t, given its initial value. (In his original papers, [1,2], Bohm introduced as fundamental, the equation of motion for the acceleration $d\vec{v}/dt$. This equation can be obtained by taking the time derivative of Equation 1, but it is misleading to regard it as *fundamental*, because it implies that the initial velocity of the particle can be assigned arbitrarily. But given the initial position $\vec{q}$, this velocity is determined uniquely by Equation 1. Bohm's equation of motion leads to the appearance of a non-local "quantum potential" that accounts for the origin of an acceleration even when the classical potential vanishes.). But it turns out that the product $m\vec{v}_B$ is not equal to the canonical momentum $\vec{p}$, because $\vec{v}_B$ does not correspond to the velocity $\vec{v}$, that is determined in quantum mechanics by the operator

$$\vec{v} = -\frac{\iota\hbar}{m}\vec{\nabla}_q = \frac{\vec{p}}{m}. \qquad (3)$$

A proof of this relation is given in section 5. Setting

$$\psi = R\, exp(\iota S/\hbar), \qquad (4)$$

where R is the amplitude of ψ, we obtain

$$\vec{v}\psi = (\vec{\nabla}_q S/m - \iota\hbar\vec{\nabla}_q R/mR)\psi. \qquad (5)$$

But in Bohm's definition of the particle velocity, Equation 1, only the first term on the right hand side of this equation appears. The relevance of the second term can be illustrated by considering the mean values $\langle\vec{v}\rangle$ and $\langle\vec{v}^2\rangle$ in this representation for ψ. We have

$$\langle\vec{v}\rangle = \int d^3q\,\psi^\dagger\vec{v}\psi = \int d^3q R^2\vec{\nabla}S/m = \langle\vec{v}_B\rangle, \qquad (6)$$

and

$$\langle\vec{v}^2\rangle = \int d^3q\,\psi^\dagger(\vec{v})^2\psi = \langle(\vec{v}_B)^2\rangle + (\hbar/m)^2\int d^3q\,(\vec{\nabla}R)^2. \qquad (7)$$

Hence, Equation 7 implies that the second moment of the velocity distribution in conventional quantum mechanics differs from that obtained in Bohm's interpretation of the particle velocity, Equation 1, by the appearance of the additional term $(\hbar/m)^2\langle(\vec{\nabla}R)^2/R^2\rangle$ on the right hand side of this equation. Remarkably, this discrepancy is not even mentioned in any of the recent articles on Bohm's interpretation of wave mechanics [5–15]. Similar discrepancies also appear in all the higher moments of this distribution.

3 Bohmian "osmotic velocity", "fluid flow pathlines" and particle trajectories

To get agreement with the mean value $\langle\vec{v}^2\rangle$ in quantum mechanics, Equation 7, Bohm's interpretation requires, in addition to the Bohmian particle velocity $\vec{v}_B$ given by Equation 1, the existence of an *ad hoc* random velocity

$$\vec{v}_o = \frac{\hbar}{mR}\vec{\nabla}R, \qquad (8)$$

with vanishing mean value, Originally, such a contribution was introduced with an undetermined coefficient as a *random* velocity by D. Bohm and J. P. Vigier [20], who named it an "osmotic velocity", after a term introduced by Einstein to describe the chaotic Brownian motion. But now such a term has been abandoned in discussions of Bohmian mechanics.

In particular, for stationary solutions of the Schrödinger, the phase $S = 0$, and Bohm's interpretation leads to the conclusion that the particle velocity vanishes in such a state. This conclusion is explained by invoking a *quantum force* due to a non-local *quantum potential* that supposedly balances the force due to the conventional potential that gives rise to the stationary solution. This non-classical force appears when the acceleration $d^2\vec{q}/dt^2$ is calculated by taking the time derivative of Equation 1 and Equation 2. But this result contradicts the fact that in quantum mechanics the velocity or momentum distribution for stationary solutions, given by the absolute square of the Fourier transform of ψ in coordinate space, is not a delta function at $\vec{v} = 0$, as is implied by Bohm's interpretation.

The trajectories obtained by integrating Bohm's first order differential equation for the particle coordinate $\vec{q}$, Equation 2, correspond to *pathlines* associated with the probability distribution $\rho = |\psi|^2$ which satisfies, like a normal fluid of density ρ, the continuity equation,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}_q\cdot\vec{j} = 0 \qquad (9)$$

where $\vec{j} = \vec{v}_B\rho$ is the associated current. While pathlines provide a visualization of a fluid flow, these lines do not correspond to the actual motion of the particles composing the fluid that also can have a random component. Likewise, Bohmian pathlines serve to visualize the evolution of the probability distribution in quantum mechanics, but do not correspond to actual trajectories of elementary particles.

Recently, experiments have been made with water droplets surfing on the waves produced by the Faraday instability on the surface of an oscillating tank filled with

a fluid [21]. The motion of these droplets mimics the suggestion of de Broigle and of Bohm that elementary particles are likewise "piloted" by the ψ function of wave mechanics. In particular, it is claimed that when the waves propagate through two slits, or are confined in a "corral", the droplets satisfy statistics that are similar to those observed for particles in quantum mechanics. But such experiments only demonstrate the universality of wave propagation, and the associated pathlines, whether governed by the equations of fluid mechanics, quantum mechanics, or of other sources of waves in physics.

4 Discussion

In his original articles [1,2], Bohm proposed an extension of de Broigle's pilot wave theory of quantum mechancs which he asserted to be equivalent to Schrödinger's formulation of wave mechanics (together with Born's statistical interpretation). In Bohm's theory, particles move along classical trajectories with a velocity determined by the phase of Schrödinger's wave function ψ, satisfying a second order Newtonian-like equation of motion, but with an additional force due to a so-called "quantum potential", that is obtained from a solution of Schrödinger's equation. In his theory, the statistical character emerges from the unknown initial velocity and position of the particle that is given by the probability distribution $|\psi|^2$. Actually, Bohm's velocity satisfies a *first* order equation, and therefore only the initial position of the particle, but not its velocity, can be imposed arbitrarily.

We have shown that in Bohm's interpretation of quantum mechanics, the product of mass m times the velocity v of a particle does not correspond to the momentum. Hence, this interpretation is not only inconsistent with the standard formulation of quantum mechanics, but also with classical mechanics, where momentum is *defined* by the relation $p = mv$. But such inconsistencies were not mentioned in Bohm's original articles, and are now generally ignored in the vast literature on this subject.

5 Appendix. The relation between velocity and momentum in non-relativistic quantum mechanics

In quantum mechanics, the velocity $\vec{v}$, like the position $\vec{q}$ and the momentum $\vec{p}$, is an operator. It is *defined* by the relation

$$\vec{v} = \frac{\iota}{\hbar}[H, \vec{q}\,], \qquad (10)$$

where H is the hamiltonian operator, and $[a, b] = ab - ba$ is the commutator of the operators a and b. In non-relativistic quantum mechanics,

$$H = -\frac{\hbar^2}{2m}\nabla_q^2 + V(\vec{q}), \qquad (11)$$

corresponding to the time dependent Schrödinger equation

$$\iota\hbar\frac{\partial\psi}{dt} = H\psi \qquad (12)$$

Hence, substituting this expression for H in Equation 10, one finds that the velocity operator is given by

$$\vec{v} = \frac{\vec{p}}{m} \qquad (13)$$

where

$$\vec{p} = -\iota\hbar\vec{\nabla}_q \qquad (14)$$

is the momentum operator.

For an alternative derivation of the connection between the velocity and momentum operators, Equation 13, that does not presuppose the Schrödinger equation, Equation 11 and Equation 12, consider the commutation relation Equation 10 for the Hamiltonian of a free particle $H_0 = \vec{p}^2/2m$. Then, according to the definition of velocity, Equation 10,

$$v_i = \frac{\iota}{2\hbar m}(p_j[p_j, q_i] + [p_j, q_i]p_j), \qquad (15)$$

and substituting the Heisenberg-Born commutation relation

$$[p_j, q_i] = -\iota\hbar\delta_{i,j} \qquad (16)$$

leads again to Equation 13.

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The author asserted that "when unambiguous the predictions of the two theories [Bohmian mechanics and standard quantum mechanics] are identical."

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In a departure from the usual Bohmian interpretation of quantum mechanics, the authors suggest that "if the Bohmian equations are understood as hydrodynamic equations, the trajectories obtained from the equation of motion (see Equation 1) should not be regarded necessarily as the trajectories pursued by real particles, but rather as the streamlines associated with the quantum fluid". This article also contains a large number of references to the literature on Bohm's interpretation of quantum mechanics.

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The author concluded that Bohmian mechanics "agrees completely with orthodox quantum theory in its predictions. Precise and simple it involves an almost obvious incorporation of Schrödiger's equation into an entirely deterministic reformulation of quantum theory".

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A related demonstration of the inconsistency of Bohm's interpretation of quantum mechanics was pointed out by Keller more than 60 years ago, but it has either been ignored or forgotten.

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