A Theory of Quantum Instruments

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Editors: ***James F. Glazebrook & Danko D. Georgiev***

Article history: Submitted on May 28, 2023; Accepted on June 2, 2023; Published on June 6, 2023.

Until recently, a quantum instrument was defined to be a completely positive operation-valued measure from the set of states on a Hilbert space to itself. In the last few years, this definition has been generalized to such measures between sets of states from different Hilbert spaces called the input and output Hilbert spaces. This article presents a theory of such instruments. Ways that instruments can be combined such as convex combinations, post-processing, sequential products, tensor products and conditioning are studied. We also consider marginal, reduced instruments and how these are used to define coexistence (compatibility) of instruments. Finally, we present a brief introduction to quantum measurement models where the generalization of instruments is essential. Many of the concepts of the theory are illustrated by examples. In particular, we discuss Holevo and Kraus instruments.

Quanta 2023; 12: 27–40.

1 Introduction

In classical physics a measurement of a physical system does not alter the state of the system. Because of this, a measurement does not interfere with later measurements. An important characteristic of quantum mechanics is that the state of a system can change into an updated state when a measurement is performed. An even more surprising and radical possibility has been recently introduced [1–4]. These works have pointed out that when the initial state ρ of a quantum system is represented by a density operator on an input Hilbert space H, then the updated state after a measurement is performed may be represented by a density operator ρ_1 in a different output Hilbert space H_1. Not only can the state of the system change as the result of a measurement, but the entire system can be altered so it is described by a different Hilbert space. This is truly an amazing new possibility! In this work we represent measurements by instruments acting on states of a Hilbert space. We present a theory of quantum instruments that emphasizes this new possibility.

Ways that instruments can be combined such as convex combinations, post-processing, tensor products, sequential products and conditioning are studied [5–9]. We also consider marginal and reduced instruments. These concepts are employed to define coexistence (compatibility) of instruments and observables. Although compatibility has been well presented in the literature [1–4, 10], we point out some of its features here. Even when two instruments have different output spaces, if their input space H is the same, then the observables they measure are on H. Because of this, we can compare these measured observables. Finally, we consider measurement models that can be used to measure instruments [11, 12]. These models strongly rely on the fact that instruments can have different input and output spaces. Many of the concepts of the theory are illustrated by examples. In particular, a theory of Holevo and Kraus instruments are considered [13–15].

Section 2 presents the basic concepts and definitions of the theory. In particular, we discuss the concepts of effects, observables, operations and instruments [5, 11, 12, 16, 17]. Section 3 gives examples of various instruments that illustrate the theory. An important role is played by Holevo and Kraus instruments [13–15]. In Section 4, we discuss theorems and results concerning instruments and observables. For example, we show that an observable conditioned on an instrument coexists with the observable measured by the instrument. Section 5 introduces the concept of a quantum measurement model. The instrument that such a model measures employs a Lüders instrument [18]. We also give a new definition of the sequential product of measurement models [5].

2 Basic Definitions and Concepts

In this work, all of our Hilbert spaces are assumed to be finite dimensional. Although this is a strong restriction, it is general enough to include theories of quantum computation and information [11, 12]. We retain this restriction for mathematical simplicity even though many of our results can be extended to the infinite dimensional case. The set of (bounded) linear operators on a Hilbert space H is denoted by $\mathcal{L}(H)$ and the zero and identity operators are 0 and I, respectively. When it is necessary to distinguish the Hilbert space, we write I_H instead of I. An *operation* from H to H_1 is a completely positive, trace non-increasing, linear map $\mathcal{J}\colon\mathcal{L}(H)\to\mathcal{L}(H_1)$ [11, 12, 17]. We denote the set of operations from H to H_1 by $\mathcal{O}(H,H_1)$. For simplicity, we write $\mathcal{O}(H)=\mathcal{O}(H,H)$ when $H=H_1$. If $\mathcal{J}_1\in\mathcal{O}(H,H_1)$, $\mathcal{J}_2\in\mathcal{O}(H_1,H_2)$, their *sequential product* $\mathcal{J}_1\circ\mathcal{J}_2\in\mathcal{O}(H,H_2)$ is given by $\mathcal{J}_1\circ\mathcal{J}_2(A)=\mathcal{J}_2\left(\mathcal{J}_1(A)\right)$. If $\mathcal{J}\in\mathcal{O}(H,H_1)$ is trace preserving we call $\mathcal{J}$ a *channel*. Every operation $\mathcal{J}\in\mathcal{O}(H,H_1)$ has the form $\mathcal{J}(A)=\sum_{i=1}^{n}J_iAJ_i^*$ where $J_i\colon H\to H_1$ is a linear operator with adjoint J_i^* and $\sum_{i=i}^{n}J_i^*J_i\le I_H$ [11, 12]. The operators J_i, $i=1,2,\dots,n$ are called *Kraus operators* for $\mathcal{J}$ [15]. We have that $\mathcal{J}$ is a channel if and only if $\sum_{i=1}^{n}J_i^*J_i=I_H$. If $\mathcal{J}\in\mathcal{O}(H,H_1)$ we define the unique *dual map* $\mathcal{J}^*\colon\mathcal{L}(H_1)\to\mathcal{L}(H)$ by $\mathrm{tr}\left[B\mathcal{J}^*(A)\right]=\mathrm{tr}\left[\mathcal{J}(B)A\right]$ for all $B\in\mathcal{L}(H)$, $A\in\mathcal{L}(H_1)$ [9]. If $\mathcal{J}$ has Kraus decomposition $\mathcal{J}(A)=\sum_{i=1}^{n}J_iAJ_i^*$ then $\mathcal{J}^*(B)=\sum_{i=1}^{n}J_i^*BJ_i$. If $\mathcal{J}$ is a channel, then $\mathcal{J}^*(I_{H_1})=I_H$ because

$$\mathrm{tr}\left[B\mathcal{J}^*(I_{H_1})\right]=\mathrm{tr}\left[\mathcal{J}(B)I_{H_1}\right]=\mathrm{tr}\left[\mathcal{J}(B)\right]=1=\mathrm{tr}(BI_H)$$

for all $B\in\mathcal{L}(H)$. A positive operator $\rho\in\mathcal{L}(H)$ with trace $\mathrm{tr}(\rho)=1$ is called a *state* on H. A state describes the condition of a quantum system and the set of states on H is denoted by $\mathcal{S}(H)$. We see that if $\rho\in\mathcal{S}(H)$ and $\mathcal{J}\in\mathcal{O}(H,H_1)$ is a channel, then $\mathcal{J}(\rho)\in\mathcal{S}(H_1)$. Also, it is easy to check that $(\mathcal{J}_1\circ\mathcal{J}_2)^*=\mathcal{J}_2^*\circ\mathcal{J}_1^*$.

A (finite) *instrument* is a finite set $\mathcal{I}=\{\mathcal{I}_x\colon x\in\Omega_\mathcal{I}\}$ where $\mathcal{I}_x\in\mathcal{O}(H,H_1)$ such that $\overline{\mathcal{I}}=\sum_{x\in\Omega_\mathcal{I}}\mathcal{I}_x$ is a channel [11, 12, 17]. An instrument is sometimes called an *operation-valued measure*. We call $\Omega_\mathcal{I}$ the *outcome space* for $\mathcal{I}$ and designate the set of instruments from H to H_1 by $\mathrm{In}(H,H_1)$. We think of $\mathcal{I}\in\mathrm{In}(H,H_1)$ as an apparatus or experiment that has outcomes $x\in\Omega_\mathcal{I}$. The probability that outcome x occurs when $\mathcal{I}$ is measured and the system is in state $\rho\in\mathcal{S}(H)$ is given by the Born rule $\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]$ [11, 12]. Since $\mathcal{I}_x$ is positive and $\overline{\mathcal{I}}$ is a channel, we have that $0\le\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\le 1$ and $\sum_{x\in\Omega_\mathcal{I}}\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]=1$ so $x\mapsto\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]$ is a probability measure on $\Omega_\mathcal{I}$. If $\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\ne 0$ and $\rho\in\mathcal{S}(H)$ is the initial state of the system, then $\mathcal{I}_x(\rho)/\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\in\mathcal{S}(H_1)$ is the *updated* state after the outcome x occurs. As pointed out in Section 1, this updated state can be in a different Hilbert space H_1 than the input space H. If $\mathcal{I}\in\mathrm{In}(H,H_1)$ we call the probability measure $\Phi_\rho^\mathcal{I}(x)=\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]$ the ρ-*distribution* of $\mathcal{I}$. As we shall see, two different instruments can have the same ρ-distribution for all $\rho\in\mathcal{S}(H)$. A *bi-instrument* $\mathcal{I}\in\mathrm{In}(H,H_1)$ is an instrument whose outcome space has the product form $\Omega_\mathcal{I}=\Omega_1\times\Omega_2$ and we write $\mathcal{I}_{xy}(\rho)$, $x\in\Omega_1$, $y\in\Omega_2$. In this case, we define the 1-*marginal* and 2-*marginal* of $\mathcal{I}$ by $\mathcal{I}_x^1(\rho)=\sum_{y\in\Omega_2}\mathcal{I}_{xy}(\rho)$ and $\mathcal{I}_y^2=\sum_{x\in\Omega_1}\mathcal{I}_{xy}(\rho)$, respectively. This gives us the three instruments $\mathcal{I},\mathcal{I}^1,\mathcal{I}^2\in\mathrm{In}(H,H_1)$. Notice that these instruments give the same channels because

$$\begin{aligned}\overline{\mathcal{I}}(\rho)&=\sum_{xy}\mathcal{I}_{xy}(\rho)\\&=\sum_x\sum_y\mathcal{I}_{xy}(\rho)\\&=\sum_x\mathcal{I}_x^1(\rho)=\overline{\mathcal{I}}^{\,1}(\rho)\end{aligned}$$

and similarly, $\overline{\mathcal{I}}(\rho)=\overline{\mathcal{I}}^{\,2}(\rho)$ for all $\rho\in\mathcal{S}(H)$.

If $\mathcal{I}\in\mathrm{In}(H,H_1)$ and $\mathcal{J}\in\mathrm{In}(H_1,H_2)$, the *sequential product of $\mathcal{I}$ then $\mathcal{J}$* is the bi-instrument $\mathcal{I}\circ\mathcal{J}\in\mathrm{In}(H,H_2)$ given by

$$(\mathcal{I}\circ\mathcal{J})_{xy}(\rho)=\mathcal{J}_y\left(\mathcal{I}_x(\rho)\right)$$

for all $\rho\in\mathcal{S}(H)$, $x\in\Omega_\mathcal{I}$, $y\in\Omega_\mathcal{J}$. Notice that $\Omega_{\mathcal{I}\circ\mathcal{J}}=\Omega_\mathcal{I}\times\Omega_\mathcal{J}$. We call the 2-marginal

$$\begin{aligned}(\mathcal{J}\mid\mathcal{I})_y(\rho)&=(\mathcal{I}\circ\mathcal{J})_y^2(\rho)\\&=\sum_x(\mathcal{I}\circ\mathcal{J})_{xy}(\rho)\\&=\sum_x\mathcal{J}_y\left(\mathcal{I}_x(\rho)\right)=\mathcal{J}_y\left(\overline{\mathcal{I}}(\rho)\right)\end{aligned}$$

the *instrument* $\mathcal{J}$ *given* (or *conditioned by* or *in the context of*) $\mathcal{I}$ and we call the 1-marginal

$$\begin{aligned}(\mathcal{I}\,\mathrm{T}\,\mathcal{J})_x(\rho) &= (\mathcal{I}\circ\mathcal{J})^1_x(\rho)\\ &= \sum_y (\mathcal{I}\circ\mathcal{J})_{xy}(\rho)\\ &= \sum_y \mathcal{J}_y\left(\mathcal{I}_x(\rho)\right) = \overline{\mathcal{J}}\left(\mathcal{I}_x(\rho)\right)\end{aligned}$$

the *instrument* $\mathcal{I}$ *then* $\mathcal{J}$ [6, 9]. If $\mathcal{K}\in\mathrm{In}(H, H_1\otimes H_2)$ we have the *reduced instruments* $\mathcal{K}_1\in\mathrm{In}(H,H_1)$, $\mathcal{K}_2\in\mathrm{In}(H,H_2)$ given by the partial traces $\mathcal{K}_{1x}(\rho) = \mathrm{tr}_{H_2}\left[\mathcal{K}_x(\rho)\right]$, $\mathcal{K}_{2x}(\rho)=\mathrm{tr}_{H_1}\left[\mathcal{K}_x(\rho)\right]$. Notice that $\mathcal{K}_1,\mathcal{K}_2$ have the same ρ-distributions for all $\rho\in\mathcal{S}(H)$.

If $\mathcal{I}_i\in\mathrm{In}(H,H_1)$, $i=1,2,\ldots,n$, with the same outcome space Ω and $\lambda_i\in[0,1]$ with $\sum\limits_{i=1}^n\lambda_i = 1$, then $\mathcal{I}=\sum\limits_{i=1}^n\lambda_i\mathcal{I}_i$ given by $\mathcal{I}_x=\sum\limits_{i=1}^n\lambda_i\mathcal{I}_{ix}$, $x\in\Omega$, is called a *convex combination* of the $\mathcal{I}_i$ [7]. We have that

$$\begin{aligned}\Phi_\rho^{\sum\lambda_i\mathcal{I}_i}(x) &= \mathrm{tr}\left[\sum_{i=1}^n\lambda_i\mathcal{I}_{ix}(\rho)\right]\\ &= \sum_{i=1}^n\lambda_i\mathrm{tr}\left[\mathcal{I}_{ix}(\rho)\right] = \sum_{i=1}^n\lambda_i\Phi_x^{\mathcal{I}_i}(\rho)\end{aligned}$$

for all $\rho\in\mathcal{S}(H)$. Thus, the distribution of a convex combination is the convex combination of the distributions. Convex combinations are an important way of combining instruments. We now consider another important way. If $\mathcal{I}\in\mathrm{In}(H,H_1)$ and $\lambda_{xz}\in[0,1]$ with $\sum\limits_z\lambda_{xz}=1$ for all $x\in\Omega_{\mathcal{I}}$, then the instrument $\mathcal{P}\in\mathrm{In}(H,H_1)$ given by $\mathcal{P}_z(\rho)=\sum\limits_x\lambda_{xz}\mathcal{I}_x(\rho)$ is called a *post-processing* of $\mathcal{I}$ [1, 11]. Two instruments $\mathcal{I}\in\mathrm{In}(H,H_1)$ and $\mathcal{J}\in\mathrm{In}(H,H_2)$ *coexist* (*are compatible*) [10], denoted by $\mathcal{I}\,\mathrm{co}\,\mathcal{J}$, if there exists a *joint bi-instrument* $\mathcal{K}\in\mathrm{In}(H,H_1\otimes H_2)$ with $\Omega_{\mathcal{K}}=\Omega_{\mathcal{I}}\times\Omega_{\mathcal{J}}$ such that for all $x\in\Omega_{\mathcal{I}}$, $y\in\Omega_{\mathcal{J}}$, $\rho\in\mathcal{S}(H)$ we have

$$\mathcal{K}^1_{1x}(\rho)=\sum_{y\in\Omega_{\mathcal{J}}}\mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right]=\mathcal{I}_x(\rho)$$
$$\mathcal{K}^2_{2y}(\rho)=\sum_{x\in\Omega_{\mathcal{I}}}\mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right]=\mathcal{J}_y(\rho)$$

Thus, two coexisting instruments can be constructed from the same bi-instrument so they are simultaneously measurable. A complete discussion of this concept is found in [1–4].

Lemma 1. If $\mathcal{I}\,\mathrm{co}\,\mathcal{J}$ and $\mathcal{P}$ is a post-processing of $\mathcal{I}$, then $\mathcal{P}\,\mathrm{co}\,\mathcal{J}$.

Proof. Suppose $\mathcal{I}\in\mathrm{In}(H,H_1)$ and $\mathcal{J}\in\mathrm{In}(H,H_2)$ and let $\mathcal{K}\in\mathrm{In}(H,H_1\otimes H_2)$ be a joint bi-instrument for $\mathcal{I},\mathcal{J}$. If $\mathcal{P}_z=\sum\limits_x\lambda_{xz}\mathcal{I}_x$ is a post-processing of $\mathcal{I}$, define the bi-instrument $\mathcal{L}_{zy}=\sum\limits_x\lambda_{xz}\mathcal{K}_{xy}$. We then obtain

$$\begin{aligned}\mathcal{L}^1_{1z}(\rho) &= \sum_y\mathrm{tr}_{H_2}\left[\mathcal{L}_{zy}(\rho)\right]=\sum_{x,y}\lambda_{xz}\mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right]\\ &= \sum_x\lambda_{xz}\mathcal{K}^1_{1z}(\rho)=\sum_x\lambda_{xz}\mathcal{I}_x(\rho)=\mathcal{P}_z(\rho)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^2_{2y}(\rho) &= \sum_z\mathrm{tr}_{H_1}\left[\mathcal{L}_{zy}(\rho)\right]=\sum_{x,z}\lambda_{xz}\mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right]\\ &= \sum_x\mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right]=\mathcal{K}^2_{2y}(\rho)=\mathcal{J}_y(\rho)\end{aligned}$$

Hence, $\mathcal{L}$ is a joint bi-instrument for $\mathcal{P}$ and $\mathcal{J}$ so $\mathcal{P}\,\mathrm{co}\,\mathcal{J}$. □

If $A,B\in\mathcal{L}(H)$ satisfy $\langle\phi,A\phi\rangle\le\langle\phi,B\phi\rangle$ for all $\phi\in H$ we write $A\le B$ and if $0\le a\le I$ we call a an *effect*. An effect corresponds to a true-false (*yes-no*) experiment and $0,I$ are the effects that are always false or always true, respectively. We denote the set of effects on H by $\mathcal{E}(H)$. If $\rho\in\mathcal{S}(H)$, $a\in\mathcal{E}(H)$, the *ρ-probability* of a is $\mathrm{tr}(\rho a)$. Thus, $\mathrm{tr}(\rho a)$ is the probability that a is true (has result *yes*) when the system is in state ρ. If a is true, then its *complement* $a'=I-a\in\mathcal{E}(H)$ is false. An *observable* is a finite set of effects $A=\{A_x\colon x\in\Omega_A\}$, $A_x\in\mathcal{E}(H)$, that satisfies $\sum\limits_{x\in\Omega_A}A_x=I$. We call Ω_A the *outcome space* for A and denote the set of observables on H by $\mathrm{Ob}(H)$. An observable is also called a *positive operator-valued measure* (POVM) [11, 12, 17]. If $\rho\in\mathcal{S}(H)$ the *ρ-probability distribution* of $A\in\mathrm{Ob}(H)$ is given by $\Phi^A_\rho(x)=\mathrm{tr}(\rho A_x)$, $x\in\Omega_A$. The observable *measured* by $\mathcal{I}\in\mathrm{In}(H,H_1)$ is the unique $\widehat{\mathcal{I}}\in\mathrm{Ob}(H)$ satisfying $\mathrm{tr}(\rho\widehat{\mathcal{I}}_x)=\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]$ for all $\rho\in\mathcal{S}(H)$. Since $\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]=\mathrm{tr}\rho\mathcal{I}^*_x(I_{H_1})$ we see that $\widehat{\mathcal{I}}_x=\mathcal{I}^*_x(I_{H_1})$ for all $x\in\Omega_{\mathcal{I}}=\Omega_{\widehat{\mathcal{I}}}$. We also have the distribution

$$\Phi^{\widehat{\mathcal{I}}}_\rho(x)=\mathrm{tr}(\rho\widehat{\mathcal{I}}_x)=\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]=\Phi^{\mathcal{I}}_\rho(x)$$

for all $x\in\Omega_{\mathcal{I}}=\Omega_{\widehat{\mathcal{I}}}$. Although an instrument measures a unique observable, as we shall see, an observable is measured by many instruments.

Let $A,B\in\mathrm{Ob}(H)$ and suppose $\mathcal{I}\in\mathrm{In}(H,H_1)$ with $\widehat{\mathcal{I}}=A$. We define the *$\mathcal{I}$-sequential product of A then B* to be the observable $A\left[\mathcal{I}\right]B\in\mathrm{Ob}(H)$ given by

$$(A\left[\mathcal{I}\right]B)_y=\sum_x\mathcal{I}^*_x(B_y)$$

As with instruments a *bi-observable* is an observable of the form

$$A=\left\{A_{xy}\colon(x,y)\in\Omega_1\times\Omega_2\right\}$$

If $B \in \mathrm{Ob}(H)$, $\mathcal{I} \in \mathrm{In}(H, H_1)$, we define B *given* $\mathcal{I}$ to be the bi-observable $(B \mid \mathcal{I})_{xy} = \mathcal{I}_x^*(B_y)$. We then have $(A\,[\mathcal{I}]\,B)_y = \sum_x (B \mid \mathcal{I})_{xy}$. Two observables $A, B \in \mathrm{Ob}(H)$ *coexist*, denoted $A \,\mathrm{co}\, B$, if there exists a *joint* bi-observable $C \in \mathrm{Ob}(H)$ with marginals $C_x^1 = \sum_y C_{xy} = A_x$ and $C_y^2 = \sum_x C_{xy} = B_y$ [1–4, 10, 11]

Lemma 2. (i) If $\mathcal{I} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H_1, H_2)$, then $(\mathcal{I} \circ \mathcal{J})^* = \mathcal{J}^* \circ \mathcal{I}^*$. (ii) If $\mathcal{I} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H_1, H_2)$ and $\mathcal{I} \,\mathrm{co}\, \mathcal{J}$, then $\widehat{\mathcal{I}} \,\mathrm{co}\, \widehat{\mathcal{J}}$. (iii) Let $\mathcal{I} \in \mathrm{In}(H, H_1)$ be a convex combination $\mathcal{I} = \sum \lambda_i \mathcal{I}_i$. Then $\overline{\mathcal{I}} = \sum_i \lambda_i \overline{\mathcal{I}}_i$ and $\left(\sum_i \lambda_i \mathcal{I}_i\right)^\wedge = \sum_i \lambda_i \widehat{\mathcal{I}}_i$.

Proof. (i) For all $\rho \in \mathcal{S}(H)$, $T \in \mathcal{L}(H_2)$ we have

$$\begin{aligned}
\mathrm{tr}\left[\rho \mathcal{J}^* \circ \mathcal{I}^*(T)\right] &= \mathrm{tr}\left[\rho \mathcal{I}^*\left(\mathcal{J}^*(T)\right)\right] \\
&= \mathrm{tr}\left[\mathcal{I}(\rho)\mathcal{J}^*(T)\right] = \mathrm{tr}\left[\mathcal{J}\left(\mathcal{I}(\rho)\right)T\right] \\
&= \mathrm{tr}\left[(\mathcal{I} \circ \mathcal{J})(\rho)T\right] = \mathrm{tr}\left[\rho(\mathcal{I} \circ \mathcal{J})^*(T)\right]
\end{aligned}$$

and the result follows.
(ii) Since $\mathcal{I} \,\mathrm{co}\, \mathcal{J}$, there exists a bi-instrument $\mathcal{K} \in \mathrm{In}(H, H_1 \otimes H_2)$ such that $\mathcal{K}_{1x}^1 = \mathcal{I}_x$, $\mathcal{K}_{2y}^2 = \mathcal{J}_y$. Define the bi-observable $C_{xy} \in \mathrm{Ob}(H)$ by $C_{xy} = \widehat{\mathcal{K}}_{xy}$. Then for all $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}
\mathrm{tr}\left(\rho \sum_y C_{xy}\right) &= \mathrm{tr}\left(\rho \sum_y \widehat{\mathcal{K}}_{xy}\right) = \sum_y \mathrm{tr}\left[\mathcal{K}_{xy}(\rho)\right] \\
&= \sum_y \mathrm{tr}\left[\mathrm{tr}_{H_2}\left(\mathcal{K}_{xy}(\rho)\right)\right] \\
&= \mathrm{tr}\left[\mathrm{tr}_{H_2}\left(\sum_y \mathcal{K}_{xy}(\rho)\right)\right] \\
&= \mathrm{tr}\left[\mathcal{K}_{1x}^1(\rho)\right] = \mathrm{tr}\left[\mathcal{I}_x(\rho)\right] = \mathrm{tr}(\rho \widehat{\mathcal{I}}_x)
\end{aligned}$$

Hence, $\sum_y C_{xy} = \widehat{\mathcal{I}}_x$ and similarly $\sum_x C_{xy} = \widehat{\mathcal{J}}_y$ so $\widehat{\mathcal{I}} \,\mathrm{co}\, \widehat{\mathcal{J}}$.
(iii) We have that

$$\overline{\mathcal{I}} = \sum_x \mathcal{I}_x = \sum_x \sum_i \lambda_i \mathcal{I}_{ix} = \sum_i \lambda_i \sum_x \mathcal{I}_{ix} = \sum_i \lambda_i \overline{\mathcal{I}}_i$$

Moreover, for all $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}
\mathrm{tr}\left[\rho\left(\sum_i \lambda_i \mathcal{I}_i\right)^\wedge\right] &= \mathrm{tr}\left[\sum_i \lambda_i \mathcal{I}_i(\rho)\right] = \sum_i \lambda_i \mathrm{tr}\left[\mathcal{I}_i(\rho)\right] \\
&= \sum_i \lambda_i(\rho \widehat{\mathcal{I}}_i) = \mathrm{tr}\left(\rho \sum_i \lambda_i \widehat{\mathcal{I}}_i\right)
\end{aligned}$$

so $\left(\sum_i \lambda_i \mathcal{I}_i\right)^\wedge = \sum_i \lambda_i \widehat{\mathcal{I}}_i$. □

For a bi-instrument $\mathcal{K} \in \mathrm{In}(H, H_1 \otimes H_2)$ we defined the marginals $\mathcal{K}_1^1$ and $\mathcal{K}_2^2$. We also have the *mixed marginals* $\mathcal{K}_{1x}^2, \mathcal{K}_{2y}^1$ given by

$$\mathcal{K}_{1y}^2(\rho) = \sum_{x \in \Omega_\mathcal{I}} \mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right]$$

$$\mathcal{K}_{2x}^1(\rho) = \sum_{y \in \Omega_\mathcal{J}} \mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right]$$

Example 1. The simplest example of an instrument is a *trivial instrument* $\mathcal{J} \in \mathrm{In}(H, H_2)$ given by $\mathcal{J}_y(\rho) = \beta_y$ for all $\rho \in \mathcal{S}(H)$, where $\beta_y \in \mathcal{E}(H_2)$ with $\beta = \sum \beta_y \in \mathcal{S}(H_2)$. Then $\mathcal{I} \,\mathrm{co}\, \mathcal{J}$ for all $\mathcal{I} \in \mathrm{In}(H, H_1)$. Indeed, let $\mathcal{K} \in \mathrm{In}(H, H_1 \otimes H_2)$ be the bi-instrument $\mathcal{K}_{xy}(\rho) = \mathcal{I}_x(\rho) \otimes \beta_y$, $x \in \Omega_\mathcal{I}$. Then for all $\rho \in \mathcal{S}(H)$ we have

$$\begin{aligned}
\mathcal{K}_{1x}^1(\rho) &= \sum_y \mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right] \\
&= \sum_y \mathrm{tr}_{H_2}\left[\mathcal{I}_x(\rho) \otimes \beta_y\right] = \mathcal{I}_x(\rho) \\
\mathcal{K}_{2y}^2(\rho) &= \sum_x \mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right] \\
&= \sum_x \mathrm{tr}_{H_1}\left[\mathcal{I}_x(\rho) \otimes \beta_y\right] = \beta_y = \mathcal{J}_y(\rho)
\end{aligned}$$

Hence, $\mathcal{K}$ is a joint instrument for $\mathcal{I}$ and $\mathcal{J}$ so $\mathcal{I} \,\mathrm{co}\, \mathcal{J}$. Notice that the mixed marginals of $\mathcal{K}$ become:

$$\begin{aligned}
\mathcal{K}_{1y}^2(\rho) &= \sum_x \mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right] = \sum_x \mathrm{tr}_{H_2}\left[\mathcal{I}_x(\rho) \otimes \beta_y\right] \\
&= \mathrm{tr}_{H_2}\left[\overline{\mathcal{I}}(\rho) \otimes \beta_y\right] = \mathrm{tr}(\beta_y)\overline{\mathcal{I}}(\rho) \\
\mathcal{K}_{2x}^1(\rho) &= \sum_y \mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right] = \sum_y \mathrm{tr}_{H_1}\left[\mathcal{I}_x(\rho) \otimes \beta_y\right] \\
&= \mathrm{tr}\left[\mathcal{I}_x(\rho)\right] \sum_y \beta_y = \mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\beta
\end{aligned}$$

We also have $\overline{\mathcal{J}}(\rho) = \beta$ for all $\rho \in \mathcal{S}(H)$ and since

$$\mathrm{tr}(\rho \widehat{\mathcal{J}}_y) = \mathrm{tr}\left[\mathcal{J}_y(\rho)\right] = \mathrm{tr}(\beta_y) = \mathrm{tr}\left[\rho \mathrm{tr}(\beta_x) I_H\right]$$

we obtain $\widehat{\mathcal{J}}_y = \mathrm{tr}(\beta_y) I_H$. We call $\widehat{\mathcal{J}}_y$ an *identity observable* [7].

Let $\mathcal{J} \in \mathrm{In}(H, H_1)$ be a trivial instrument with $\mathcal{J}_x(\rho) = \beta_x$, $\beta_x \in \mathcal{E}(H_1)$. If $\mathcal{I} \in \mathrm{In}(H_1, H_2)$ is arbitrary, we have the sequential product $\mathcal{J} \circ \mathcal{I} \in \mathrm{In}(H_1, H_2)$ given by

$$(\mathcal{J} \circ \mathcal{I})_{xy}(\rho) = \mathcal{I}_y\left(\mathcal{J}_x(\rho)\right) = \mathcal{I}_y(\beta_x)$$

We then have $\overline{\mathcal{J} \circ \mathcal{I}}(\rho) = \overline{\mathcal{I}}(\beta)$ for all $\rho \in \mathcal{S}(H)$. Since

$$\begin{aligned}
\mathrm{tr}\left[\rho(\mathcal{J} \circ \mathcal{I})_{xy}^\wedge\right] &= \mathrm{tr}\left[(\mathcal{J} \circ \mathcal{I})_{xy}(\rho)\right] \\
&= \mathrm{tr}\left[\mathcal{I}_y(\beta_x)\right] \\
&= \mathrm{tr}\left[\rho \mathrm{tr}\left(\mathcal{I}_y(\beta_x)\right) I_H\right]
\end{aligned}$$

we obtain $(\mathcal{J}\circ\mathcal{I})^\wedge_{xy}=\mathrm{tr}\left[\mathcal{I}_y(\beta_x)\right]I_H$ which is an identity bi-observable. The conditional instrument $(\mathcal{I}\mid\mathcal{J})\in\mathrm{In}(H_1,H_2)$ becomes

$$(\mathcal{I}\mid\mathcal{J})_y(\rho)=\mathcal{I}_y\left(\overline{\mathcal{J}}(\rho)\right)=\mathcal{I}_y(\beta)$$

for all $\rho\in\mathcal{S}(H)$. If $\mathcal{I}\in\mathrm{In}(H_0,H)$ is arbitrary, we have the sequential product $\mathcal{I}\circ\mathcal{J}\in\mathrm{In}(H_0,H_1)$ given by

$$(\mathcal{I}\circ\mathcal{J})_{xy}(\rho)=\mathcal{J}_y\left(\mathcal{I}_x(\rho)\right)=\mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\beta_y$$

We then have $\overline{\mathcal{I}\circ\mathcal{J}}(\rho)=\beta$ for all $\rho\in\mathcal{S}(H_0)$. Since

$$\begin{aligned}\mathrm{tr}\left[\rho(\mathcal{I}\circ\mathcal{J})^\wedge_{xy}\right]&=\mathrm{tr}\left[(\mathcal{I}\circ\mathcal{J})_{xy}(\rho)\right]\\&=\mathrm{tr}\left[\mathcal{I}_x(\rho)\mathrm{tr}(\beta_y)\right]\\&=\mathrm{tr}(\rho\widehat{\mathcal{I}}_x)\mathrm{tr}(\beta_y)\\&=\mathrm{tr}\left[\rho\,\mathrm{tr}(\beta_y)\widehat{\mathcal{I}}_x\right]\end{aligned}$$

we obtain $(\mathcal{I}\circ\mathcal{J})^\wedge_{xy}=\mathrm{tr}(\beta_y)\widehat{\mathcal{I}}_x$. The conditional instrument $(\mathcal{J}\mid\mathcal{I})\in\mathrm{In}(H_0,H_1)$ becomes

$$(\mathcal{J}\mid\mathcal{I})_y(\rho)=\mathcal{J}_y\left(\overline{\mathcal{I}}(\rho)\right)=\beta_y=\mathcal{J}_y(\rho)$$

so $(\mathcal{J}\mid\mathcal{I})=\mathcal{J}$. □

If $A\in\mathrm{Ob}(H_1)$, $B\in\mathrm{Ob}(H_2)$, define the *tensor product bi-observable* $A\otimes B\in\mathrm{Ob}(H_1\otimes H_2)$ by $(A\otimes B)_{xy}=A_x\otimes B_y$ [7]. We then have $(A\otimes B)^1_x=A_x\otimes I_{H_2}$, $(A\otimes B)^2_y=I_{H_1}\otimes B_y$ and the identity observables $(A\otimes B)^1_{2x}=\mathrm{tr}(A_x)I_{H_2}$, $(A\otimes B)^2_{1y}=\mathrm{tr}(B_y)I_{H_1}$. Now $A\otimes B$ is a joint bi-observable for A,B in the sense that $\frac{1}{n_2}(A\otimes B)^1_{1x}=A_x$ and $\frac{1}{n_1}(A\otimes B)^2_{2y}=B_y$ where $n_2=\dim H_2$, $n_1=\dim H_1$.

If $\mathcal{I}\in\mathcal{O}(H_1,H_3)$, $\mathcal{J}\in\mathcal{O}(H_2,H_4)$, define the *tensor product* $\mathcal{K}=\mathcal{I}\otimes\mathcal{J}$ to be the operation $\mathcal{K}\in(H_1\otimes H_2,H_3\otimes H_4)$ that satisfies

$$\mathcal{K}(C\otimes D)=\mathcal{I}(C)\otimes\mathcal{J}(D)$$

for all $C\in\mathcal{L}(H_1)$, $D\in\mathcal{L}(H_2)$. To show that $\mathcal{K}$ exists, suppose $\mathcal{I}$ and $\mathcal{J}$ have Kraus decompositions $\mathcal{I}(C)=\sum_i K_iCK_i^*$, $\mathcal{J}(D)=\sum_j J_jDJ_j^*$ where $\sum_i K_i^*K_i\le I_{H_1}$, $\sum_j J_j^*J_j\le I_{H_2}$. Then for $E\in\mathcal{L}(H_1\otimes H_2)$ we define

$$\mathcal{K}(E)=\sum_{i,j}K_i\otimes J_jEK_i^*\otimes J_j^*$$

Then

$$\begin{aligned}\sum_{i,j}(K_i^*\otimes J_j^*)(K_i\otimes J_j)&=\sum_{i,j}(K_i^*K_i\otimes J_j^*J_j)\\&=\sum_i K_i^*K_i\otimes\sum_j J_j^*J_j\\&\le I_{H_1}\otimes I_{H_2}\end{aligned}$$

and $\mathcal{K}\in\mathcal{O}(H_1\otimes H_2,H_3\otimes H_4)$ satisfies

$$\begin{aligned}K(C\otimes D)&=\sum_{i,j}K_i\otimes J_jC\otimes DK_i^*\otimes J_j^*\\&=\sum_{i,j}(K_iCK_i^*)\otimes(J_jDJ_j^*)\\&=\sum_i K_iCK_i^*\otimes\sum_j J_jDJ_j^*=\mathcal{I}(C)\otimes\mathcal{J}(D)\end{aligned}$$

for all $C\in\mathcal{L}(H_1)$, $D\in\mathcal{L}(H_2)$.

If $\mathcal{I}\in\mathrm{In}(H_1,H_3)$, $J\in\mathrm{In}(H_2,H_4)$ define the *tensor product* $\mathcal{K}=\mathcal{I}\otimes\mathcal{J}$ to be the bi-instrument $\mathcal{K}\in\mathrm{In}(H_1\otimes H_2,H_3\otimes H_4)$ defined by $\mathcal{K}_{xy}(\rho)=\mathcal{I}_x\otimes\mathcal{J}_y(\rho)$ for all $\rho\in\mathcal{S}(H_1\otimes H_2)$. We have seen that $\mathcal{K}_{xy}\in\mathcal{O}(H_1\otimes H_2,H_3\otimes H_4)$ and $\overline{\mathcal{K}}$ is a channel because $\overline{\mathcal{K}}=\overline{\mathcal{I}}\otimes\overline{\mathcal{J}}$ and $\overline{\mathcal{I}},\overline{\mathcal{J}}$ are channels. The next result shows that $\mathcal{I}\otimes\mathcal{J}$ is a type of joint instrument for $\mathcal{I},\mathcal{J}$.

Theorem 3. Let $\mathcal{I}\in\mathrm{In}(H_1,H_3)$, $\mathcal{J}=\mathrm{In}(H_2,H_4)$ and let $\mathcal{K}=\mathcal{I}\otimes\mathcal{J}$. (i) $\widehat{\mathcal{K}}_{xy}=\widehat{\mathcal{I}}_x\otimes\widehat{\mathcal{J}}_y$. (ii) For all $\rho\in\mathcal{S}(H_1\otimes H_2)$ we have $\mathcal{K}^1_{1x}(\rho)=\mathcal{I}_x\left[\mathrm{tr}_{H_2}(\rho)\right]$, $\mathcal{K}^2_{2y}(\rho)=\mathcal{J}_y\left[\mathrm{tr}_{H_1}(\rho)\right]$. (iii) If $n_1=\dim H_1$, $n_2=\dim H_2$, $\rho_1\in\mathcal{S}(H_1)$, $\rho_2=\mathcal{S}(H_2)$ we have

$$\begin{aligned}\tfrac{1}{n_2}\mathcal{K}^1_{1x}(\rho_1\otimes I_{H_2})&=\mathcal{I}_x(\rho_1)\\\tfrac{1}{n_1}\mathcal{K}^2_{2y}(I_{H_1}\otimes\rho_2)&=\mathcal{J}_y(\rho_2)\end{aligned}$$

Proof. (i) For all $\rho=\rho_1\otimes\rho_2\in\mathcal{L}(H_1\otimes H_2)$ we have

$$\begin{aligned}\mathrm{tr}(\rho\widehat{\mathcal{K}}_{xy})&=\mathrm{tr}\left[\mathcal{K}_{xy}(\rho)\right]=\mathrm{tr}\left[\mathcal{I}_x\otimes\mathcal{J}_y(\rho_1\otimes\rho_2)\right]\\&=\mathrm{tr}\left[\mathcal{I}_x(\rho_1)\otimes\mathcal{J}_y(\rho_2)\right]=\mathrm{tr}\left[\mathcal{I}_x(\rho_1)\right]\mathrm{tr}\left[\mathcal{J}_y(\rho_2)\right]\\&=\mathrm{tr}(\rho_1\widehat{\mathcal{I}}_x)\mathrm{tr}(\rho_2\widehat{\mathcal{J}}_y)=\mathrm{tr}(\rho_1\widehat{\mathcal{I}}_x\otimes\rho_2\widehat{\mathcal{J}}_y)\\&=\mathrm{tr}(\rho_1\otimes\rho_2\widehat{\mathcal{I}}_x\otimes\widehat{\mathcal{J}}_y)=\mathrm{tr}(\rho\widehat{\mathcal{I}}_x\otimes\widehat{\mathcal{J}}_y)\end{aligned}$$

Since any $A\in\mathcal{L}(H_1\otimes H_2)$ has the form $A=\sum_{i,j}B_i\otimes C_j$, $B_i\in\mathcal{L}(H_1)$, $C_j\in\mathcal{L}(H_2)$, the result holds for $\rho=A$. Hence, $\widehat{\mathcal{K}}_{xy}=\widehat{\mathcal{I}}_x\otimes\widehat{\mathcal{J}}_y$.

(ii) For all $\rho=\rho_1\otimes\rho_2\in\mathcal{L}(H_1\otimes H_2)$ we have

$$\begin{aligned}\mathcal{K}^1_{1x}(\rho)&=\mathrm{tr}_{H_4}\left[\sum_y\mathcal{K}_{xy}(\rho)\right]\\&=\mathrm{tr}_{H_4}\left[\sum_y\mathcal{I}_x\otimes\mathcal{J}_y(\rho_1\otimes\rho_2)\right]\\&=\mathrm{tr}_{H_4}\left[\sum_y\mathcal{I}_x(\rho_1)\otimes\mathcal{J}_y(\rho_2)\right]\\&=\sum_y\mathrm{tr}_{H_4}\left[\mathcal{I}_x(\rho_1)\otimes\mathcal{J}_y(\rho_2)\right]\\&=\mathrm{tr}_{H_4}\left[\mathcal{I}_x(\rho_1)\otimes\overline{\mathcal{J}}(\rho_2)\right]=\mathcal{I}_x(\rho_1)=\mathcal{I}_x\left[\mathrm{tr}_{H_2}(\rho)\right]\end{aligned}$$

As in (i) the result follows for all $\rho \in \mathcal{S}(H_1 \otimes H_2)$.
(iii) Applying (i) we obtain

$$\begin{aligned}\mathcal{K}^1_{1x}(\rho_1 \otimes I_{H_2}) &= \mathcal{I}_x\left[\mathrm{tr}_{H_2}(\rho_1 \otimes I_{H_2})\right] \\ &= \mathcal{I}_x\left[\mathrm{tr}(I_{H_2})\rho_1\right] \\ &= n_2\mathcal{I}_x(\rho_1)\end{aligned}$$

Hence, $\frac{1}{n_2}\mathcal{K}^1_{1x}(\rho_1 \otimes I_{H_2}) = \mathcal{I}(\rho_1)$. Similarly, $\frac{1}{n_1}\mathcal{K}^2_{2y}(I_{H_1} \otimes \rho_2) = \mathcal{J}_y(\rho_2)$. □

3 Examples of Instruments

Two important instruments are the Holevo and Kraus instruments. These instruments are useful for illustrating the definitions and concepts presented in Section 2. If $A \in \mathrm{Ob}(H)$ and $\alpha = \{\alpha_x \colon x \in \Omega_A\} \subseteq \mathcal{S}(H_1)$, the corresponding *Holevo instrument* $\mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_1)$ has the form $\mathcal{H}^{(A,\alpha)}_x(\rho) = \mathrm{tr}(\rho A_x)\alpha_x$ for all $\rho \in \mathcal{S}(H)$ [6, 13, 14]. Notice that $\mathcal{H}^{(A,\alpha)}$ is indeed an instrument because

$$\begin{aligned}\sum_x \mathrm{tr}(\rho A_x) &= \mathrm{tr}\left(\rho\sum_x A_x\right) \\ &= \mathrm{tr}(\rho) = 1\end{aligned}$$

for every $\rho \in \mathcal{S}(H)$ so $\sum_x \mathrm{tr}(\rho A_x)\alpha_x$ is a convex combination of states which is a state. Since

$$\begin{aligned}\mathrm{tr}\left[\rho\mathcal{H}^{(A,\alpha)*}_x(a)\right] &= \mathrm{tr}\left[\mathcal{H}^{(A,\alpha)}_x(\rho)a\right] \\ &= \mathrm{tr}\left[\mathrm{tr}(\rho A_x)\alpha_x a\right] \\ &= \mathrm{tr}\left[\rho\,\mathrm{tr}(\alpha_x a)A_x\right]\end{aligned}$$

we have that $\mathcal{H}^{(A,\alpha)*}_x(a) = \mathrm{tr}(\alpha_x a)A_x$ for all $a \in \mathcal{E}(H_1)$. We conclude thhat

$$(\mathcal{H}^{(A,\alpha)}_x)^\wedge = \mathcal{H}^{(A,\alpha)*}_x(I_{H_1}) = A_x$$

so $\mathcal{H}^{(A,\alpha)\wedge} = A$. We also have $\overline{\mathcal{H}}^{(A,\alpha)}(\rho) = \sum\limits_x \mathrm{tr}(\rho A_x)\alpha_x$ which, as we showed previously is a state.

If $\mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_1)$ and $\mathcal{H}^{(B,\beta)} \in \mathrm{In}(H_1, H_2)$, then their sequential product becomes

$$\begin{aligned}\left[\mathcal{H}^{(A,\alpha)} \circ \mathcal{H}^{(B,\beta)}\right]_{xy}(\rho) &= \mathcal{H}^{(B,\beta)}_y\left[\mathcal{H}^{(A,\alpha)}_x(\rho)\right] \\ &= \mathcal{H}^{(B,\beta)}_y\left[\mathrm{tr}(\rho A_x)\alpha_x\right] \\ &= \mathrm{tr}(\rho A_x)\mathcal{H}^{(B,\beta)}_y(\alpha_x) \\ &= \mathrm{tr}(\rho A_x)\mathrm{tr}(\alpha_x B_y)\beta_y \\ &= \mathrm{tr}\left[\rho\,\mathrm{tr}(\alpha_x B_y)A_x\right]\beta_y \\ &= \mathcal{H}^{(C_y\beta)}_{xy}(\rho)\end{aligned}$$

We conclude that $\mathcal{H}^{(A,\alpha)} \circ \mathcal{H}^{(B,\beta)} = \mathcal{H}^{(C,\beta)}$ where $C \in \mathrm{Ob}(H)$ is the bi-observable given by $C_{xy} = \mathrm{tr}(\alpha_x B_y)A_x$. The conditioned instrument $(\mathcal{H}^{(B,\beta)} \mid \mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_2)$ becomes

$$\begin{aligned}(\mathcal{H}^{(B,\beta)} \mid \mathcal{H}^{(A,\alpha)})_y(\rho) &= \mathcal{H}^{(B,\beta)}_y\left[\overline{\mathcal{H}^{(A,\alpha)}}\rho\right] \\ &= \mathcal{H}^{(B\beta)}_y\left[\sum_x \mathcal{H}^{(A,\alpha)}_x(\rho)\right] \\ &= \sum_x \mathcal{H}^{(B,\beta)}_y\left[\mathcal{H}^{(A,\alpha)}_x(\rho)\right] \\ &= \sum_x \mathcal{H}^{(C,\beta)}_{xy}(\rho) = \mathcal{H}^{(C,\beta)2}_y(\rho)\end{aligned}$$

We conclude that $(\mathcal{H}^{(B,\beta)} \mid \mathcal{H}^{(A,\alpha)})$ is the marginal instrument $\mathcal{H}^{(C,\beta)2}$. We also have

$$\begin{aligned}(\mathcal{H}^{(A,\alpha)}\,\mathrm{T}\,\mathcal{H}^{(B,\beta)})_x(\rho) &= \overline{\mathcal{H}^{(B,\beta)}}\left[\mathcal{H}^{(A,\alpha)}_x(\rho)\right] \\ &= \sum_y \mathcal{H}^{(B,\beta)}_y\left[\mathcal{H}^{(A,\alpha)}_x(\rho)\right] \\ &= \sum_y \mathcal{H}^{(C,\beta)}_{xy}(\rho) = \mathcal{H}^{(C,\beta)1}_x(\rho)\end{aligned}$$

Hence, $(\mathcal{H}^{(A,\alpha)}\,\mathrm{T}\,\mathcal{H}^{(B,\beta)})$ is the marginal instrument $\mathcal{H}^{(C,\beta)1}$. Notice that $C^1_x = A_x$ so $C^1 = A$ and

$$C^2_y = \sum_x \mathrm{tr}(\alpha_x B_y)A_x$$

Since $\sum\limits_x \mathrm{tr}(\alpha_x B_y) = 1$ for every $y \in \Omega_B$, C^2 is a post-processing of A.

Let $A_{xy} \in \mathrm{Ob}(H)$ be a bi-observable, $\alpha = \left\{\alpha_{xy} \colon (x,y) \in \Omega_A\right\} \subseteq \mathcal{S}(H_1 \otimes H_2)$ and define the Holevo bi-instrument in $\mathrm{In}(H, H_1 \otimes H_2)$ by

$$\mathcal{H}^{(A,\alpha)}_{xy}(\rho) = \mathrm{tr}(\rho A_{xy})\alpha_{xy}$$

The marginals become

$$\begin{aligned}\mathcal{H}^{(A,\alpha)1}_{xy}(\rho) &= \sum_y \mathcal{H}^{(A,\alpha)}_{xy} = \sum_y \mathrm{tr}(\rho A_{xy})\alpha_{xy} \\ \mathcal{H}^{(A,\alpha)2}_{xy}(\rho) &= \sum_x \mathcal{H}^{(A,\alpha)}_{xy} = \sum_x \mathrm{tr}(\rho A_{xy})\alpha_{xy}\end{aligned}$$

We then have the reduced and mixed marginals

$$\begin{aligned}\mathcal{H}^{(A,\alpha)1}_{1x}(\rho) &= \sum_y \mathrm{tr}(\rho A_{xy})\mathrm{tr}_{H_2}(\alpha_{xy}) \in \mathrm{In}(H, H_1) \\ \mathcal{H}^{(A,\alpha)2}_{2y}(\rho) &= \sum_x \mathrm{tr}(\rho A_{xy})\mathrm{tr}_{H_1}(\alpha_{xy}) \in \mathrm{In}(H, H_2) \\ \mathcal{H}^{(A,\alpha)2}_{1y}(\rho) &= \sum_x \mathrm{tr}(\rho A_{xy})\mathrm{tr}_{H_2}(\alpha_{xy}) \in \mathrm{In}(H, H_1) \\ \mathcal{H}^{(A,\alpha)1}_{2x}(\rho) &= \sum_y \mathrm{tr}(\rho A_{xy})\mathrm{tr}_{H_1}(\alpha_{xy}) \in \mathrm{In}(H, H_2)\end{aligned}$$

We say that $\mathcal{H}^{(A,\alpha)}$ is a *product instrument* if $\alpha_{xy} = \beta_x \otimes \gamma_y$, $\beta_x \in \mathcal{S}(H_1)$, $\gamma_y \in \mathcal{S}(H_2)$ and in this case we have

$$\mathcal{H}^{(A,\alpha)1}_{1x}(\rho) = \sum_y \mathrm{tr}(\rho A_{xy})\beta_x$$

$$\mathcal{H}_{2y}^{(A,\alpha)2}(\rho) = \sum_x \mathrm{tr}(\rho A_{xy})\gamma_y$$

Notice that $\mathcal{H}_{1x}^{(A,\alpha)1} = \mathcal{H}_x^{(B,\beta)}$ where $B_x = \sum_y A_{xy} = A_x^1$ and $\mathcal{H}_{2y}^{(A,\alpha)2} = \mathcal{H}_y^{(C,\gamma)}$ where $C_y = \sum_x A_{xy} = A_y^2$.

Suppose $\mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_1)$, $\mathcal{H}^{(B,\beta)} \in \mathrm{In}(H, H_2)$ and $\mathcal{H}^{(A,\alpha)}$ so $\mathcal{H}^{(B,\beta)}$. If their joint instrument is $\mathcal{H}^{(C,\gamma)} \in \mathrm{In}(H, H_1 \otimes H_2)$ then for all $\rho \in \mathcal{S}(H)$ we have

$$\begin{aligned}
\mathrm{tr}(\rho A_x)\alpha_x &= \mathcal{H}_x^{(A,\alpha)}(\rho) = \mathcal{H}_{1x}^{(C,\gamma)1} \\
&= \sum_y \mathrm{tr}(\rho C_{xy})\mathrm{tr}_{H_2}(\gamma_{xy}) \\
\mathrm{tr}(\rho B_y)\beta_y &= \mathcal{H}_y^{(B,\beta)}(\rho) = \mathcal{H}_{2y}^{(C,\gamma)2} \\
&= \sum_x \mathrm{tr}(\rho C_{xy})\mathrm{tr}_{H_1}(\gamma_{xy})
\end{aligned}$$

If C is a product instrument with $\gamma_{xy} = \varepsilon_x \otimes \delta_y$ we obtain

$$\begin{aligned}
\mathrm{tr}(\rho A_x)\alpha_x &= \sum_y \mathrm{tr}(\rho C_{xy})\varepsilon_x \\
&= \mathrm{tr}\left(\rho \sum_y C_{xy}\right)\varepsilon_x \\
&= \mathrm{tr}(\rho C_x^1)\varepsilon_x \\
\mathrm{tr}(\rho B_y)\beta_y &= \sum_x \mathrm{tr}(\rho C_{xy})\delta_y \\
&= \mathrm{tr}\left(\rho \sum_x C_{xy}\right)\delta_y \\
&= \mathrm{tr}(\rho C_y^2)\delta_y
\end{aligned}$$

It follows that $\varepsilon_x = \alpha_x$, $A_x = C_x^1$ and $\beta_y = \delta_y$, $B_y = C_y^2$. Moreover, $\gamma_{xy} = \alpha_x \otimes \beta_y$.

A *Kraus instrument* $\mathcal{K} \in \mathrm{In}(H, H_1)$ has the form $\mathcal{K}_x(\rho) = K_x \rho K_x^*$ where $K_x \colon \mathcal{L}(H) \to \mathcal{L}(H_1)$ are linear operators satisfying $\sum_x K_x^* K_x = I_H$ [15]. We call K_x the *Kraus operators* for $\mathcal{K}$. Notice that $0 \le K_x^* K_x \le I_H$ so $K_x^* K_x \in \mathcal{E}(H)$ for all $x \in \Omega_{\mathcal{K}}$. Since

$$\mathrm{tr}\left[\mathcal{K}_x(\rho)a\right] = \mathrm{tr}(K_x \rho K_x^* a) = \mathrm{tr}(\rho K_x^* a K_x)$$

for every $a \in \mathcal{L}(H_1)$ we have $\mathcal{K}_x^*(a) = K_x^* a K_x$. It follows that the measured observable $\widehat{\mathcal{K}} \in \mathrm{Ob}(H)$ is

$$\widehat{\mathcal{K}}_x = \mathcal{K}_x^*(I_{H_1}) = K_x^* K_x$$

for all $x \in \Omega_{\mathcal{K}}$. Let $\mathcal{K} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H_1, H_2)$ be Kraus instruments with operators K_x, J_y, respectively. Then $\mathcal{K} \circ \mathcal{J} \in \mathrm{In}(H, H_2)$ is the bi-instrument given by

$$\begin{aligned}
(\mathcal{K} \circ \mathcal{J})_{xy}(\rho) &= \mathcal{J}_y\left[\mathcal{K}_x(\rho)\right] = J_y(K_x \rho K_x^*)J_y^* \\
&= J_y K_x \rho (J_y K_x)^* = \mathcal{L}_{xy}(\rho)
\end{aligned}$$

where $\mathcal{L}_{xy}$ is the Kraus bi-instrument with Kraus operators $L_{xy} = J_y K_x$. It follows that $(\mathcal{J} \mid \mathcal{K}) \in \mathrm{In}(H, H_2)$ is given by

$$\begin{aligned}
(\mathcal{J} \mid \mathcal{K})_y(\rho) &= \mathcal{J}_y\left(\overline{\mathcal{K}}(\rho)\right) = \mathcal{J}_y\left(\sum_x K_x \rho K_x^*\right) \\
&= \sum_x \left[\mathcal{J}_y(K_x \rho K_x^*)\right] = \sum_x (J_y K_x \rho K_x^* J_y^*) \\
&= \sum_x \mathcal{L}_{xy}(\rho) = \mathcal{L}_y^2(\rho)
\end{aligned}$$

We also have

$$\begin{aligned}
(\mathcal{K} \,\mathrm{T}\, \mathcal{J})_x(\rho) &= \overline{\mathcal{J}}\left[\mathcal{K}_x(\rho)\right] = \sum_y \mathcal{J}_y(K_x \rho K_x^*) \\
&= \sum_y J_y K_x \rho K_x^* J_y^* \\
&= \sum_y \mathcal{L}_{xy}(\rho) = \mathcal{L}_x^1(\rho)
\end{aligned}$$

Let $\mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H_1, H_2)$ be Holevo and $\mathcal{K} \in \mathrm{In}(H, H_1)$ be an arbitrary instrument. We then have the bi-instrument $\mathcal{K} \circ \mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_2)$ as follows

$$\begin{aligned}
(\mathcal{K} \circ \mathcal{H}^{(A,\alpha)})_{xy}(\rho) &= \mathcal{H}_y^{(A,\alpha)}(\mathcal{K}_x(\rho)) = \mathrm{tr}\left[\mathcal{K}_x(\rho)A_y\right]\alpha_y \\
&= \mathrm{tr}\left[\rho \mathcal{K}_x^*(A_y)\right]\alpha_y = \mathcal{H}_{xy}^{(B,\alpha)}(\rho)
\end{aligned}$$

where $B \in \mathrm{Ob}(H)$ is the bi-observable given by $B_{xy} = \mathcal{K}_x^*(A_y)$. We conclude that

$$(\mathcal{K} \circ \mathcal{H}^{(A,\alpha)})_{xy}^{\wedge} = B_{xy} = \mathcal{K}_x^*(A_y)$$

We also have

$$\begin{aligned}
(\mathcal{H}^{(A,\alpha)} \mid \mathcal{K})_y(\rho) &= \mathcal{H}_y^{(A,\alpha)}(\overline{\mathcal{K}}(\rho)) = \mathcal{H}_y^{(A,\alpha)}\left[\sum_x \mathcal{K}_x(\rho)\right] \\
&= \mathrm{tr}\left[\rho \sum_x \mathcal{K}_x^*(A_y)\right]\alpha_y = \mathrm{tr}(\rho B_y^2)\alpha_y \\
&= \mathcal{H}_y^{(B^2,\alpha)}(\rho)
\end{aligned}$$

Hence, $(\mathcal{H}^{(A,\alpha)} \mid \mathcal{K}) = \mathcal{H}^{(B^2,\alpha)}$ which is Holevo. Moreover,

$$\begin{aligned}
(\mathcal{K} \,\mathrm{T}\, \mathcal{H}^{(A,\alpha)})_x(\rho) &= \overline{\mathcal{H}^{(A,\alpha)}}\left[\mathcal{K}_x(\rho)\right] \\
&= \sum_y \mathrm{tr}\left[\mathcal{K}_x(\rho)A_y\right]\alpha_y \\
&= \sum_y \mathrm{tr}\left[\rho \mathcal{K}_x^*(A_y)\right]\alpha_y \\
&= \sum_y \mathrm{tr}\left[\rho B_{xy}\right]\alpha_y \\
&= \sum_y \mathcal{H}_{xy}^{(B,\alpha)}(\rho) = \mathcal{H}_x^{(B,\alpha)1}(\rho)
\end{aligned}$$

Therefore, $(\mathcal{K}\,\mathrm{T}\,H^{(A,\alpha)}) = \mathcal{H}^{(B,\alpha)1}$ which is a marginal of a Holevo bi-instrument. We conclude that the sequential product of an arbitrary instrument then a Holevo instrument is Holevo and a Holevo instrument conditioned by an arbitrary instrument is Holevo. In particular, if $\mathcal{K}$ is Kraus with operators K_x, then $\mathcal{K}\circ\mathcal{H}^{(A,\alpha)} = \mathcal{H}^{(B,\alpha)}$ where $B_{xy} = K_x^*A_yK_x$.

In the other order, let $\mathcal{H}^{(A,\alpha)} \in \mathrm{In}(H, H_1)$ and $\mathcal{K} \in \mathrm{In}(H_1, H_2)$ be arbitrary. Then $\mathcal{H}^{(A,\alpha)} \circ \mathcal{K} \in \mathrm{In}(H, H_2)$ is the bi-instrument given by

$$\begin{aligned}(\mathcal{H}^{(A,\alpha)} \circ \mathcal{K})_{xy}(\rho) &= \mathcal{K}_y\left[\mathcal{H}_x^{(A,\alpha)}(\rho)\right]\\ &= \mathcal{K}_y\left[\mathrm{tr}(\rho A_x)\alpha_x\right]\\ &= \mathrm{tr}(\rho A_x)\mathcal{K}_y(\alpha_x)\end{aligned}$$

If $\mathcal{K}_y(\alpha_x) \neq 0$, let $\beta_{xy} \in \mathcal{S}(H_2)$ be defined by $\beta_{xy} = \mathcal{K}_y(\alpha_x)/\mathrm{tr}\left[\mathcal{K}_y(\alpha_y)\right]$ and define the bi-observable $B_{xy} = \mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]A_x$. We then obtain

$$\begin{aligned}(\mathcal{H}^{(A,\alpha)} \circ \mathcal{K})_{xy}(\rho) &= \mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]\mathrm{tr}(\rho A_x)\beta_{xy}\\ &= \mathrm{tr}(\rho B_{xy})\beta_{xy} = \mathcal{H}_{xy}^{(B,\beta)}(\rho)\end{aligned}$$

which is a Holevo bi-instrument. Hence,

$$(\mathcal{H}^{(A,\alpha)} \circ \mathcal{K})^{\wedge}_{xy} = B_{xy} = \mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]A_x$$

We also have

$$\begin{aligned}(\mathcal{K} \mid \mathcal{H}^{(A,\alpha)})_y(\rho) &= \mathcal{K}_y(\overline{\mathcal{H}^{(A,\alpha)}}(\rho))\\ &= \mathcal{K}_y\left[\sum_x \mathcal{H}_x^{(A,\alpha)}(\rho)\right]\\ &= \sum_x \mathcal{K}_y\left[\mathrm{tr}(\rho A_x)\alpha_x\right]\\ &= \sum_x \mathrm{tr}(\rho A_x)\mathcal{K}_y(\alpha_x)\\ &= \sum_x \mathrm{tr}(\rho A_x)\mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]\beta_{xy}\\ &= \sum_x \mathrm{tr}(\rho B_{xy})\beta_{xy}\\ &= \sum_x \mathcal{H}_{xy}^{(B,\beta)}(\rho) = \mathcal{H}_y^{(B,\beta)2}(\rho)\end{aligned}$$

Therefore, $(\mathcal{K} \mid \mathcal{H}^{(A,\alpha)}) = \mathcal{H}^{(B,\beta)2}$ which is a marginal of a Holevo bi-instrument. Moreover,

$$\begin{aligned}(\mathcal{H}^{(A,\alpha)}\,\mathrm{T}\,\mathcal{K})_x(\rho) &= \overline{\mathcal{K}}\left[\mathcal{H}_x^{(A,\alpha)}(\rho)\right]\\ &= \sum_y \mathcal{K}_y\left[\mathrm{tr}(\rho A_x)\alpha_x\right]\\ &= \mathrm{tr}(\rho A_x)\sum_y \mathcal{K}_y(\alpha_x)\\ &= \mathrm{tr}(\rho A_x)\sum_y \mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]\beta_{xy}\\ &= \sum_y \mathrm{tr}\left\{\rho\,\mathrm{tr}\left[\mathcal{K}_y(\alpha_x)\right]A_x\right\}\beta_{xy}\\ &= \sum_y \mathrm{tr}(\rho B_{xy})\beta_{xy}\\ &= \sum_y \mathcal{H}_{xy}^{(B,\beta)}(\rho) = \mathcal{H}_x^{(B,\beta)1}(\rho)\end{aligned}$$

Hence, $(\mathcal{H}^{(A,\alpha)}\,\mathrm{T}\,\mathcal{K}) = \mathcal{H}^{(B,\beta)1}$ which is a marginal of a Holevo bi-instrument.

We now give an example of a convex tensor product of two instruments. Let $\mathcal{I} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H, H_2)$, $\alpha_x \in \mathcal{S}(H_1)$, $\beta_y \in \mathcal{S}(H_2)$, $\lambda_y, \mu_x \in [0, 1]$ with $\sum_y \lambda_y + \sum_x \mu_x = 1$ and define $\lambda = \sum\limits_y \lambda_y$, $\mu = \sum\limits_x \mu_x$. Define the bi-instrument $\mathcal{K} \in \mathrm{In}(H, H_1 \otimes H_2)$ by

$$\mathcal{K}_{xy}(\rho) = \lambda_y\mathcal{I}_x(\rho) \otimes \beta_y + \mu_x\alpha_x \otimes \mathcal{J}_y(\rho)$$

Notice that $\mathcal{K}$ is indeed an instrument because

$$\begin{aligned}\mathrm{tr}\left[\sum_{x,y}\mathcal{K}_{xy}(\rho)\right] &= \sum_{x,y}\mathrm{tr}\left[\mathcal{K}_{xy}(\rho)\right]\\ &= \sum_{x,y}\left\{\lambda_y\mathrm{tr}\left[\mathcal{I}_x(\rho)\right] + \mu_x\mathrm{tr}\left[\mathcal{J}_y(\rho)\right]\right\}\\ &= \sum_y \lambda_y\mathrm{tr}\left[\overline{\mathcal{I}}(\rho)\right] + \sum_x \mu_x\mathrm{tr}\left[\overline{\mathcal{J}}(\rho)\right]\\ &= \sum_y \lambda_y + \sum_x \mu_x = 1\end{aligned}$$

The marginals $\mathcal{K}^1 \in \mathrm{In}(H, H_1 \otimes H_2)$, $\mathcal{K}^2 \in \mathrm{In}(H, H_1 \otimes H_2)$ are given by

$$\begin{aligned}(\mathcal{K}_x^1(\rho) &= \sum_y \mathcal{K}_{xy}(\rho) = \mathcal{I}_x(\rho) \otimes \sum_y \lambda_y\beta_y + \mu_x\alpha_x \otimes \overline{\mathcal{J}}(\rho)\\ \mathcal{K}_y^2(\rho) &= \sum_x \mathcal{K}_{xy}(\rho) = \overline{\mathcal{I}}(\rho) \otimes \lambda_y\beta_y + \sum_x \mu_x\alpha_x \otimes \mathcal{J}_y(\rho)\end{aligned}$$

The reduced instruments $K_1 \in \mathrm{In}(H, H_1)$, $\mathcal{K}_2 \in \mathrm{In}(H, H_2)$ become

$$\begin{aligned}\mathcal{K}_{1xy}(\rho) &= \mathrm{tr}_{H_2}\left[\mathcal{K}_{xy}(\rho)\right] = \lambda_y\mathcal{I}_x(\rho) + \mu_x\mathrm{tr}\left[\mathcal{J}_y(\rho)\right]\alpha_x\\ \mathcal{K}_{2xy}(\rho) &= \mathrm{tr}_{H_1}\left[\mathcal{K}_{xy}(\rho)\right] = \lambda_y\mathrm{tr}\left[\mathcal{J}_x(\rho)\right]\beta_y + \mu_x\mathcal{J}_y(\rho)\end{aligned}$$

The reduced marginals $K_1^1 \in \mathrm{In}(H, H_1)$, $\mathcal{K}_2^2 \in \mathrm{In}(H, H_2)$ $\mathcal{K}_1^2 \in \mathrm{In}(H, H_1)$, $\mathcal{K}_2^1 \in \mathrm{In}(H, H_2)$ are given by

$$\begin{aligned}\mathcal{K}_{1x}^1(\rho) &= \sum_y \mathcal{K}_{1xy}(\rho) = \lambda\mathcal{I}_x(\rho) + \mu_x\alpha_x\\ \mathcal{K}_{2y}^2(\rho) &= \sum_x \mathcal{K}_{2xy}(\rho) = \lambda_y\beta_y + \mu\mathcal{J}_y(\rho)\\ \mathcal{K}_{1y}^2(\rho) &= \sum_x \mathcal{K}_{1xy}(\rho) = \lambda_y\overline{\mathcal{I}}(\rho) + \mathrm{tr}\left[\mathcal{J}_y(\rho)\right]\sum_x \mu_x\alpha_x\\ \mathcal{K}_{2x}^1(\rho) &= \sum_y \mathcal{K}_{2xy}(\rho) = \mathrm{tr}\left[\mathcal{I}_x(\rho)\right]\sum_y \lambda_y\beta_y + \mu_x\overline{\mathcal{J}}(\rho)\end{aligned}$$

We have that $\mathcal{K}_1^1 \operatorname{co} \mathcal{K}_2^2$ and $\mathcal{K}_1^2 \operatorname{co} \mathcal{K}_2^1$. The measured observables are gotten as follows:

$$\begin{aligned}\operatorname{tr}(\rho\widehat{\mathcal{K}}_{xy}) &= \operatorname{tr}\left[\mathcal{K}_{xy}(\rho)\right] \\ &= \lambda_y \operatorname{tr}\left[\mathcal{I}_x(\rho)\right] + \mu_x \operatorname{tr}\left[\mathcal{J}_y(\rho)\right] \\ &= \lambda_y \operatorname{tr}(\rho\widehat{\mathcal{I}}_x) + \mu_x \operatorname{tr}(\rho\widehat{\mathcal{J}}_y)\end{aligned}$$

Hence, $\widehat{\mathcal{K}}_{xy} = \lambda_y\widehat{\mathcal{I}}_x + \mu_x\widehat{\mathcal{J}}_y$. Therefore, $\widehat{\mathcal{K}}_x^1 = \lambda\widehat{\mathcal{I}}_x + \mu_x I_H$ and $\widehat{\mathcal{K}}_y^2 = \lambda_y I_H + \mu\widehat{\mathcal{J}}_y$ coexist with joint observable $\widehat{\mathcal{K}}_{xy}$. We also have

$$\begin{aligned}\operatorname{tr}(\rho\widehat{\mathcal{K}}_{1x}^1) &= \operatorname{tr}\left[\mathcal{K}_{1x}^1(\rho)\right] \\ &= \lambda \operatorname{tr}\left[\mathcal{I}_x(\rho)\right] + \mu_x \\ &= \lambda \operatorname{tr}(\rho\widehat{\mathcal{I}}_x) + \mu_x \operatorname{tr}(\rho) \\ &= \operatorname{tr}\left[\rho(\lambda\widehat{\mathcal{I}}_x + \mu_x I_H)\right]\end{aligned}$$

Hence, $\widehat{\mathcal{K}}_{1x}^1 = \lambda\widehat{\mathcal{I}}_x + \mu_x I_H = \widehat{\mathcal{K}}_x^1$ and similarly $\widehat{\mathcal{K}}_{2y}^2 = \lambda_y I_H + \mu\widehat{\mathcal{J}}_y = \widehat{\mathcal{K}}_y^2$. Moreover,

$$\begin{aligned}\operatorname{tr}(\rho\widehat{\mathcal{K}}_{1y}^2) &= \operatorname{tr}\left[\mathcal{K}_{1y}^2(\rho)\right] = \lambda_y + \mu \operatorname{tr}\left[\mathcal{J}_y(\rho)\right] \\ &= \operatorname{tr}\left[\rho(\mu\widehat{\mathcal{J}}_y + \lambda_y I_H)\right]\end{aligned}$$

Therefore,

$$\widehat{\mathcal{K}}_{1y}^2 = \mu\widehat{\mathcal{J}}_y + \lambda_u I_H = \widehat{\mathcal{K}}_{2y}^2 = \widehat{\mathcal{K}}_y^2$$

and similarly $\widehat{\mathcal{K}}_{2x}^1 = \widehat{\mathcal{K}}_{1x}^1 = \widehat{\mathcal{K}}_x^1$.

Let us consider the special case in which $\mathcal{I} = \mathcal{H}^{(A,\gamma)}$ and $\mathcal{J} = \mathcal{H}^{(B,\delta)}$. We then obtain

$$\begin{aligned}\mathcal{K}_{xy}(\rho) &= \lambda_y \mathcal{H}_x^{(A,\gamma)}(\rho) \otimes \beta_y + \mu_x \alpha_x \otimes \mathcal{H}_y^{(B,\delta)}(\rho) \\ &= \lambda_y \operatorname{tr}(\rho A_x)\gamma_x \otimes \beta_y + \mu_x \alpha_x \otimes \operatorname{tr}(\rho B_y)\gamma_y\end{aligned}$$

In this case, we have

$$\begin{aligned}\mathcal{K}_{1xy}(\rho) &= \lambda_y \operatorname{tr}(\rho A_x)\gamma_x + \mu_x \operatorname{tr}(\rho B_y)\alpha_x \\ \mathcal{K}_{2xy}(\rho) &= \lambda_y \operatorname{tr}(\rho A_x)\beta_y + \mu_x \operatorname{tr}(\rho B_y)\delta_y\end{aligned}$$

We also obtain $\widehat{\mathcal{K}}_{xy} = \lambda_y A_x + \mu_x B_y$, $\widehat{\mathcal{K}}_x^1 = \lambda A_x + \mu_x I_H$, $\widehat{\mathcal{K}}_y^2 = \lambda_y I_H + \mu B_y$.

4 Results

Our first result shows that a convex combination of Holevo instruments with the same base Hilbert space, outcome space and states is Holevo. Moreover, a weakened form of the converse holds.

Theorem 4. (i) Let $\mathcal{H}^{(A_i,\alpha)}$, $i = 1, 2, \ldots, n$, be Holevo instruments in $\operatorname{In}(H, H_1)$ with the same outcome space Ω and states $\alpha = \{\alpha_x : x \in \Omega\}$. Then a convex combination $\sum_{i=1}^{n} \lambda_i \mathcal{H}^{(A_i,\alpha)}$ is Holevo and

$$\sum_{i=1}^{n} \lambda_i \mathcal{H}^{(A_i,\alpha)} = \mathcal{H}^{(\sum \lambda_i A_i,\alpha)}$$

(ii) If $\mathcal{H}^{(A_i,\alpha_i)} \in \operatorname{In}(H, H_1)$ with the same outcomes space Ω and if

$$\sum_{i=1}^{n} \lambda_i \mathcal{H}^{(A_i,\alpha_i)} = \mathcal{H}^{(B,\beta)}$$

then $B = \sum \lambda_i A_i$ and

$$\beta_x = \frac{1}{\sum\limits_i \lambda_i \operatorname{tr}(A_{ix})} \sum_i \lambda_i \operatorname{tr}(A_{ix})\alpha_{ix} \tag{1}$$

for all $x \in \Omega$.

Proof. (i) For all $x \in \Omega$, we obtain

$$\begin{aligned}\sum_i \lambda_i \mathcal{H}_x^{(A_i,\alpha)}(\rho) &= \sum_i \lambda_i \operatorname{tr}(\rho A_{ix})\alpha_x \\ &= \operatorname{tr}\left[\rho\left(\sum_i \lambda_i A_i\right)_x\right]\alpha_x \\ &= \mathcal{H}_x^{(\sum \lambda_i A_i,\alpha)}(\rho)\end{aligned}$$

and the result follows.

(ii) For all $\rho \in \mathcal{S}(H)$ and $x \in \Omega$ we have

$$\begin{aligned}\operatorname{tr}(\rho B_x)\beta_x = \mathcal{H}_x^{(B,\beta)}(\rho) &= \sum_i \lambda_i \mathcal{H}_x^{(A_i,\alpha_i)}(\rho) \\ &= \sum_i \lambda_i \operatorname{tr}(\rho A_{ix})\alpha_{ix}\end{aligned} \tag{2}$$

Taking the trace of (2) gives

$$\operatorname{tr}(\rho B_x) = \sum_i \lambda_i \operatorname{tr}(\rho A_{ix}) = \operatorname{tr}\left(\rho \sum_i \lambda_i A_{ix}\right)$$

Hence, $B_x = \sum\limits_i \lambda_i A_{ix}$ for all $x \in \Omega$ and we conclude that $B = \sum\limits_i \lambda_i A_i$. Substituting B into (2) gives

$$\sum_i \lambda_i \operatorname{tr}(\rho A_{ix})\beta_x = \sum_i \lambda_i \operatorname{tr}(\rho A_{ix})\alpha_{ix}$$

so that

$$\beta_x = \frac{1}{\sum\limits_i \lambda_i \operatorname{tr}(\rho A_{ix})} \sum_i \lambda_i \operatorname{tr}(\rho A_{ix})\alpha_{ix}$$

for all $x \in \Omega$, $\rho \in \mathcal{S}(H)$. Letting $\rho = I/n$ where $n = \dim H$, we conclude that (1) holds. □

We have seen that a convex combination of Holevo instruments $\mathcal{H}^{(A_i,\alpha)}$ is Holevo. We now show that a general convex combination of Holevo instruments $\mathcal{H}^{(A_i,\alpha_i)}$ need not be Holevo.

Example 2. Let $\mathcal{H}^{(A,\alpha)}$, $\mathcal{H}^{(B,\beta)} \in \text{In}(\mathbb{C}^2)$ be Holevo instruments with the same outcome space $\Omega = \{x, y\}$ and let $A_x = B_y = |\phi\rangle\langle\phi|$ where $\phi \in \mathbb{C}^2$ with $\|\phi\| = 1$. Also, assume that $\alpha_x \neq \beta_x$ and

$$\tfrac{1}{2}\mathcal{H}^{(A,\alpha)} + \tfrac{1}{2}\mathcal{H}^{(B,\beta)} = \mathcal{H}^{(C,\gamma)}$$

It follows from Theorem 4(ii) that $C = \frac{1}{2}A + \frac{1}{2}B$ so

$$C_x = \tfrac{1}{2}A_x + \tfrac{1}{2}B_x = \tfrac{1}{2}I = C_y$$

Also from Theorem 4(ii) we obtain $\gamma_x = \frac{1}{2}(\alpha_x + \beta_x)$. Since

$$\tfrac{1}{2}\text{tr}(\rho A_x)\alpha_x + \tfrac{1}{2}\text{tr}(\rho B_x)\beta_x = \text{tr}(\rho C_x)\gamma_x$$

for all $\rho \in \mathcal{S}(\mathbb{C}^2)$, letting $\rho = A_x$ we have $\alpha_x = \gamma_x = \frac{1}{2}(\alpha_x + \beta_x)$. But then $\alpha_x = \beta_x$ which is a contradiction. Hence, $\frac{1}{2}\mathcal{H}^{(A,\alpha)} + \frac{1}{2}\mathcal{H}^{(B,\beta)}$ is not Holevo. This also shows that the converse of Theorem 4(ii) does not hold. □

Example 3. This example shows that a convex combination of Kraus instruments need not be Kraus. Let $\{\phi_1, \phi_2\}$ be an orthonormal basis for $\mathbb{C}^2$, let K_x, K_y be the projection $K_x = |\phi_1\rangle\langle\phi_1|$, $K_y = |\phi_2\rangle\langle\phi_2|$ and let $J_x = K_y$, $J_y = K_x$. Define the Kraus instruments $\mathcal{K}, \mathcal{J} \in \text{In}(\mathbb{C}^2)$ with operators $\{K_x, K_y\}$, $\{J_x, J_y\}$, respectively. Suppose $\mathcal{L} \in \text{In}(\mathbb{C}^2)$ is a Kraus instrument with outcome space $\Omega = \{x, y\}$, operators $\{L_x, L_y\}$ so that $L_x^*L_x + L_y^*L_y = I$ and $\mathcal{L} = \frac{1}{2}\mathcal{K} + \frac{1}{2}\mathcal{J}$. We then obtain

$$L_x\rho L_x^* = \mathcal{L}_x(\rho) = \tfrac{1}{2}\mathcal{K}_x(\rho) + \tfrac{1}{2}\mathcal{J}_x(\rho) = \tfrac{1}{2}K_x\rho K_x + \tfrac{1}{2}J_x\rho J_x$$

for all $\rho \in \mathcal{S}(\mathbb{C}^2)$. Letting $\rho = I/2$ we have

$$L_xL_x^* = \tfrac{1}{2}K_x + \tfrac{1}{2}J_x = \tfrac{1}{2}I$$

and it follows that $\sqrt{2}L_x$ is a unitary operator U. Hence, for all $\rho \in \mathcal{S}(\mathbb{C}^2)$ we have

$$U\rho U^* = K_x\rho K_x + J_x\rho J_x$$

Therefore,

$$K_xU\rho U^* = K_x\rho K_x = U\rho U^*K_x$$

We conclude that K_x commutes with every $\rho \in \mathcal{S}(H)$. Hence, $K_x = \lambda_x I$, $\lambda_x \in [0, 1]$ which is a contradiction. □

Lemma 5. If $\mathcal{J} \in \text{In}(H, H_1)$ is a post-processing of a Holevo instrument $\mathcal{I} \in \text{In}(H, H_1)$, then $\mathcal{J}$ is Holevo.

Proof. Suppose $\mathcal{I} = \mathcal{H}^{(A,\alpha)}$ and $\mathcal{J}$ is a post-processing of $\mathcal{I}$. Then there exist $\lambda_{xy} \in [0, 1]$ with $\sum\limits_y \lambda_{xy} = 1$ for all $x \in \Omega_{\mathcal{I}}$ such that

$$\begin{aligned}\mathcal{J}_y(\rho) &= \sum_x \lambda_{xy}\mathcal{I}_x(\rho) = \sum_x \lambda_{xy}\mathcal{H}_x^{(A,\alpha)}(\rho)\\ &= \sum_x \lambda_{xy}\text{tr}(\rho A_x)\alpha_x = \text{tr}\left(\rho \sum_x \lambda_{xy}A_x\right)\alpha_x\\ &= \mathcal{H}_y^{\left(\sum\limits_x \lambda_{xy}A_x, \alpha\right)}(\rho)\end{aligned}$$

Hence, $\mathcal{J} = \mathcal{H}^{(B,\alpha)}$ is Holevo with $B_y = \sum\limits_x \lambda_{xy}A_x$ a post-processing of A. □

We conjecture that Lemma 5 does not hold for Kraus instruments but have not found a counterexample.

Lemma 6. If $\mathcal{I} \in \text{In}(H, H_1)$, $\mathcal{J} \in \text{In}(H_1, H_2)$, $\mathcal{K} \in (H_0, H)$ and $\mathcal{I}\,\text{co}\,\mathcal{J}$, then $(\mathcal{I} \mid \mathcal{K})\,\text{co}(\mathcal{J} \mid \mathcal{K})$. If $\mathcal{L}$ is a joint instrument for $\mathcal{I}$ and $\mathcal{J}$, then $\mathcal{M} = \overline{\mathcal{K}} \circ \mathcal{L}$ is a joint instrument for $(\mathcal{I} \mid \mathcal{K})$ and $(\mathcal{J} \mid \mathcal{K})$.

Proof. Let $\mathcal{L} \in \text{In}(H, H_1 \otimes H_2)$ be a joint bi-instrument for $\mathcal{I}$, $\mathcal{J}$. Define $\mathcal{M} \in \text{In}(H_0, H_1 \otimes H_2)$ by $\mathcal{M}_{xy}(\rho) = \mathcal{L}_{xy}(\overline{\mathcal{K}}(\rho))$. We then have

$$\begin{aligned}\mathcal{M}_{1x}^1(\rho) &= \mathcal{L}_{1x}^1(\overline{\mathcal{K}}(\rho)) = \sum_{y\in\Omega_{\mathcal{J}}} \text{tr}_{H_2}\left[\mathcal{L}_{xy}(\overline{\mathcal{K}}(\rho))\right]\\ &= \mathcal{I}_x(\overline{\mathcal{K}}(\rho)) = (\mathcal{I} \mid \mathcal{K})_x(\rho)\\ \mathcal{M}_{2y}^2(\rho) &= \mathcal{L}_{2y}^2(\overline{\mathcal{K}}(\rho)) = \sum_{x\in\Omega_{\mathcal{I}}} \text{tr}_{H_1}\left[\mathcal{L}_{xy}(\overline{\mathcal{K}}(\rho))\right]\\ &= \mathcal{J}_x(\overline{\mathcal{K}}(\rho)) = (\mathcal{J} \mid \mathcal{K})_y(\rho)\end{aligned}$$

Hence, $\mathcal{M}$ is a joint bi-instrument for $(\mathcal{I} \mid \mathcal{K})$ and $(\mathcal{J} \mid \mathcal{K})$ so $(\mathcal{I} \mid \mathcal{K})\,\text{co}(\mathcal{J} \mid \mathcal{K})$. Moreover, $\mathcal{M} = \overline{\mathcal{K}} \circ \mathcal{L}$. □

If $\mathcal{I} \in \text{In}(H, H_1)$, then $\widehat{\mathcal{I}}_x = \mathcal{I}_x^*(I_{H_1}) \in \text{Ob}(H)$ and if $A \in \text{Ob}(H_1)$ we define $(A \mid \mathcal{I})_x = \overline{\mathcal{I}}^*(A_x) \in \text{Ob}(H)$. Also, if $\mathcal{J} \in \text{In}(H_1, H_2)$ then

$$(\mathcal{I} \circ \mathcal{J})_{xy}(\rho) = \mathcal{J}_y(\mathcal{I}_x(\rho)) \in \text{In}(H_1, H_2)$$

and since $(\mathcal{J} \mid \mathcal{I})_y(\rho) = \mathcal{J}_y(\overline{\mathcal{I}}(\rho))$ we have that $(\mathcal{J} \mid \mathcal{I}) \in \text{In}(H_1, H_2)$. Now $\widehat{\mathcal{J}} \in \text{Ob}(H_1)$ so

$$(\widehat{\mathcal{J}} \mid \mathcal{I})_y = \overline{\mathcal{I}}^*(\widehat{\mathcal{J}}_y) \in \text{Ob}(H)$$

Also, $(\mathcal{J} \mid \mathcal{I})^\wedge \in \text{Ob}(H)$ and the next result shows that these two observables coincide.

Lemma 7. If $\mathcal{I} \in \text{In}(H, H_1)$ and $\mathcal{J} \in \text{In}(H_1, H_2)$, then $(\mathcal{J} \mid \mathcal{I})^\wedge = (\widehat{\mathcal{J}} \mid \mathcal{I})$.

Proof. For all $y \in \Omega_{\mathcal{J}}$ and $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}\operatorname{tr}\left[\rho(\mathcal{J} \mid \mathcal{I})_y^{\wedge}\right] &= \operatorname{tr}\left[(\mathcal{J} \mid \mathcal{I})_y(\rho)\right] = \operatorname{tr}\left[\mathcal{J}_y(\overline{\mathcal{I}}(\rho))\right] \\ &= \operatorname{tr}\left[\overline{\mathcal{I}}(\rho)\widehat{\mathcal{J}}_y\right] = \operatorname{tr}\left[\rho\overline{\mathcal{I}}^*(\widehat{\mathcal{J}}_y)\right] \\ &= \operatorname{tr}\left[\rho(\widehat{\mathcal{J}} \mid \mathcal{I})_y\right]\end{aligned}$$

Hence, $(\mathcal{J} \mid \mathcal{I})^{\wedge} = (\widehat{\mathcal{J}} \mid \mathcal{I})$. □

Corollary 8. If $\mathcal{I} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H, H_2)$, $\mathcal{K} \in \mathrm{In}(H_0, H)$ and $\mathcal{I}\,\mathrm{co}\,\mathcal{J}$, then $(\widehat{\mathcal{I}} \mid \mathcal{K})\,\mathrm{co}(\widehat{\mathcal{J}} \mid \mathcal{K})$.

Proof. By Lemma 6, $(\mathcal{I} \mid \mathcal{K})\,\mathrm{co}(\mathcal{J} \mid \mathcal{K})$ so $(\mathcal{I} \mid \mathcal{K})^{\wedge}\,\mathrm{co}(\mathcal{J} \mid \mathcal{K})^{\wedge}$. By Lemma 7, $(\widehat{\mathcal{I}} \mid \mathcal{K}) = (\mathcal{I} \mid \mathcal{K})^{\wedge}$ and $(\widehat{\mathcal{J}} \mid \mathcal{K}) = (\mathcal{J} \mid \mathcal{K})^{\wedge}$ so $(\widehat{\mathcal{I}} \mid \mathcal{K})\,\mathrm{co}(\widehat{\mathcal{J}} \mid \mathcal{K})$. □

Lemma 9. Let $A, B \in \mathrm{Ob}(H)$ and $\mathcal{I} \in \mathrm{In}(H_1, H)$. If $A\,\mathrm{co}\,B$, then $(A \mid \mathcal{I})\,\mathrm{co}(B \mid \mathcal{I})$. If C is a joint bi-observable for A and B, then $D_{xy} = \overline{\mathcal{I}}^*(C_{xy})$ is a joint bi-observable for $(A \mid \mathcal{I})$ and $(B \mid \mathcal{I})$.

Proof. We have that $D, (A \mid \mathcal{I}), (B \mid \mathcal{I}) \in \mathrm{Ob}(H_1)$ and we obtain

$$\begin{aligned}D_x^1 &= \sum_y D_{xy} = \sum_y \overline{\mathcal{I}}^*(C_{xy}) \\ &= \mathcal{I}^*\left(\sum_y C_{xy}\right) = \overline{\mathcal{I}}^*(A_x) = (A \mid \mathcal{I})_x\end{aligned}$$

and similarly, $D_y^2 = (B \mid \mathcal{I})_y$. Hence, D is a joint bi-observable for $(A \mid \mathcal{I})$ and $(B \mid \mathcal{I})$ implying that $(A \mid \mathcal{I})\,\mathrm{co}(B \mid \mathcal{I})$. □

Example 4. The converse of Lemma 9 does not hold. To show this, suppose $A, B \in \mathrm{Ob}(H)$ do not coexist. Let $\mathcal{H}^{(C,\alpha)} \in \mathrm{In}(H_1, H)$ be Holevo with $C \in \mathrm{Ob}(H_1)$, $\{\alpha\} = \alpha \in \mathcal{S}(H)$. Then

$$(A \mid \mathcal{H}^{(C,\alpha)})_x = \mathcal{H}^{(C,\alpha)*}(A_x) = \sum_z \operatorname{tr}(\alpha A_x)C_z = \operatorname{tr}(\alpha A_x)I_{H_1}$$
$$(B \mid \mathcal{H}^{(C,\alpha)})_y = \mathcal{H}^{(C,\alpha)*}(B_y) = \sum_z \operatorname{tr}(\alpha B_y)C_z = \operatorname{tr}(\alpha B_y)I_{H_1}$$

Letting $D_{xy} = \operatorname{tr}(\alpha A_x)\operatorname{tr}(\alpha B_y)I_{H_1} \in \mathrm{Ob}(H_1)$, we have that D is a joint bi-observable for $(A \mid \mathcal{H}^{(C,\alpha)})$ and $(B \mid \mathcal{H}^{(C,\alpha)})$. Hence, $(A \mid \mathcal{H}^{(C,\alpha)})\,\mathrm{co}(B \mid \mathcal{H}^{(C,\alpha)})$ but A, B do not coexist. □

We say that an observable A is *sharp* if A_x is a projection for all $x \in \Omega_A$ and an instrument $\mathcal{I}$ is *sharp* if $\widehat{\mathcal{I}}$ is sharp [6, 11, 12].

Theorem 10. Let $\mathcal{I} \in \mathrm{In}(H, H_1)$ and $A \in \mathrm{Ob}(H_1)$. (i) $(A \mid \mathcal{I})\,\mathrm{co}\,\widehat{\mathcal{I}}$. (ii) If $\mathcal{I}$ is sharp, then $(A \mid \mathcal{I})$ commutes with $\widehat{\mathcal{I}}$.

Proof. (i) Let B_{xy} be the bi-observable on H given by $B_{xy} = \mathcal{I}_x^*(A_y)$. Notice that B_{xy} is indeed an observable because

$$\begin{aligned}\sum_{x,y} B_{xy} &= \sum_{x,y} \mathcal{I}_x^*(A_y) = \sum_x \mathcal{I}_x^*\left(\sum_y A_y\right) \\ &= \sum_x \mathcal{I}_x^*(I_{H_1}) = \sum_x \widehat{\mathcal{I}}_x = I_H\end{aligned}$$

We have that

$$B_x^1 = \sum_y B_{xy} = \mathcal{I}_x^*(I_{H_1}) = \widehat{\mathcal{I}}_x$$
$$B_y^2 = \sum_x B_{xy} = \sum_x \mathcal{I}_x^*(A_y) = \overline{\mathcal{I}}^*(A_y) = (A \mid \mathcal{I})_y$$

so $(A \mid \mathcal{I})\,\mathrm{co}\,\widehat{\mathcal{I}}$.

(ii) If $\mathcal{I}$ is sharp, then $\widehat{\mathcal{I}}$ is sharp and by (i) we have that $\widehat{\mathcal{I}}\,\mathrm{co}(A \mid \mathcal{I})$. It follows that $\widehat{\mathcal{I}}_x$ and $(A \mid \mathcal{I})_y$ are coexisting effects [11, 16]. Since $\widehat{\mathcal{I}}_x$ is a projection we conclude that $\widehat{\mathcal{I}}_x$ and $(A \mid \mathcal{I})_y$ commute for all x, y [11, 16]. □

Theorem 11. (i) If $\mathcal{I} \in \mathrm{In}(H, H_1)$, $\mathcal{J} \in \mathrm{In}(H_1, H_2)$, then $(\mathcal{I}_x \circ \mathcal{J}_y)^{\wedge} = \mathcal{I}_x^*(\widehat{\mathcal{J}}_y)$ for all x, y. (ii) If $\mathcal{I}, \mathcal{J} \in \mathrm{In}(H)$, then $\mathcal{I} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{I}$ implies $\mathcal{I}_x^*(\widehat{\mathcal{J}}_y) = \mathcal{J}_y^*(\widehat{\mathcal{I}}_x)$ for all x, y which implies $(\mathcal{I} \circ \mathcal{J})^{\wedge} = (\mathcal{J} \circ \mathcal{I})^{\wedge}$. (iii) If $\mathcal{I}, \mathcal{J} \in \mathrm{In}(H)$ with $\mathcal{I} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{I}$, then $(\widehat{\mathcal{I}} \mid \mathcal{J}) = \widehat{\mathcal{I}}$ and $(\widehat{\mathcal{J}} \mid \mathcal{I}) = \widehat{\mathcal{J}}$.

Proof. (i) For all $\rho \in \mathcal{S}(H)$, we have

$$\begin{aligned}\operatorname{tr}\left[\rho(\mathcal{I}_x \circ \mathcal{J}_y)^{\wedge}\right] &= \operatorname{tr}\left[\mathcal{I}_x \circ \mathcal{J}_y(\rho)\right] = \operatorname{tr}\left[\mathcal{J}_y(\mathcal{I}_x(\rho))\right] \\ &= \operatorname{tr}\left[\mathcal{I}_x(\rho)\widehat{\mathcal{J}}_y\right] = \operatorname{tr}\left[\rho\mathcal{I}_x^*(\widehat{\mathcal{J}}_y)\right]\end{aligned}$$

It follows that $(\mathcal{I}_x \circ \mathcal{J}_y)^{\wedge} = \mathcal{I}_x^*(\mathcal{J}_y)$ for x, y.

(ii) If $\mathcal{I} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{I}$, then by (i) we obtain

$$\mathcal{I}_x^*(\widehat{\mathcal{J}}_y) = (\mathcal{I}_x \circ \mathcal{J}_y)^{\wedge} = (\mathcal{J}_y \circ \mathcal{I}_x)^{\wedge} = \mathcal{J}_y^*(\widehat{\mathcal{I}}_x)$$

for all x, y. Moreover, if $\mathcal{I}_x^*(\widehat{\mathcal{J}}_y) = \mathcal{J}_y^*(\widehat{\mathcal{I}}_x)$ then by (i) we have $(\mathcal{I}_x \circ \mathcal{J}_y)^{\wedge} = (\mathcal{J}_y \circ \mathcal{I}_x)^{\wedge}$.

(iii) If $\mathcal{I} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{I}$, then by (ii) we obtain

$$\widehat{\mathcal{I}}_x = \mathcal{I}_x^*(I_H) = \sum_y \mathcal{I}_x^*(\widehat{\mathcal{J}}_y) = \sum_y \mathcal{J}_y^*(\widehat{\mathcal{I}}_x) = (\widehat{\mathcal{I}} \mid \mathcal{J})_x$$

Hence, $\widehat{\mathcal{I}} = (\widehat{\mathcal{I}} \mid \mathcal{J})$ and similarly, $\widehat{\mathcal{J}} = (\widehat{\mathcal{J}} \mid \mathcal{I})$. □

Example 5. Let $\mathcal{H}^{(A,\alpha)}, \mathcal{H}^{(B,\beta)} \in \mathrm{In}(H)$ be Holevo. We have seen in the second paragraph of Section 3 that

$$\mathcal{H}_x^{(A,\alpha)} \circ \mathcal{H}_y^{(B,\beta)}(\rho) = \operatorname{tr}(\rho A_x)\operatorname{tr}(\alpha_x B_y)\beta_y$$

and similarly,

$$\mathcal{H}_y^{(B,\beta)} \circ \mathcal{H}_x^{(A,\alpha)}(\rho) = \operatorname{tr}(\rho B_y)\operatorname{tr}(\beta_y A_x)\alpha_x$$

Hence, $\mathcal{H}_x^{(A,\alpha)} \circ \mathcal{H}_y^{(B,\beta)} = \mathcal{H}_y^{(B,\beta)} \circ \mathcal{H}_x^{(A,\alpha)}$ if and only if

$$\mathrm{tr}(\rho A_x)\mathrm{tr}(\alpha_x B_y)\beta_y = \mathrm{tr}(\rho B_y)\mathrm{tr}(\beta_y A_x)\alpha_x \tag{3}$$

for all $\rho \in \mathcal{S}(H)$. Taking the trace of (3) gives

$$\mathrm{tr}(\rho A_x)\mathrm{tr}(\alpha_x B_y) = \mathrm{tr}(\rho B_y)\mathrm{tr}(\beta_y A_x) \tag{4}$$

for all $\rho \in \mathcal{S}(H)$. Applying (4) we have

$$\mathrm{tr}\left[\rho\mathrm{tr}(\alpha_x B_y)A_x\right] = \mathrm{tr}\left[\rho\mathrm{tr}(\beta_y A_x)B_y\right] \tag{5}$$

so we have

$$\mathrm{tr}(\alpha_x B_y)A_x = \mathrm{tr}(\beta_y A_x)B_y \tag{6}$$

Applying (3) and (4) we obtain $\beta_y = \alpha_x = \gamma \in \mathcal{S}(H)$ for all x, y and (6) becomes

$$\mathrm{tr}(\gamma B_y)A_x = \mathrm{tr}(\gamma A_x)B_y$$

for all x, y. Summing over y gives $A_x = \mathrm{tr}(\gamma A_x)I_H$. We conclude that if

$$\mathcal{H}^{(A,\alpha)} \circ \mathcal{H}^{(B,\beta)} = \mathcal{H}^{(B,\beta)} \circ \mathcal{H}^{(A,\alpha)} \tag{7}$$

then $AB = BA$. The converse does not hold because we can have $AB = BA$ but (3) does not hold (for example, let $A_x \neq \mathrm{tr}(\gamma A_x)I_H$) so (7) does not hold. □

5 Measurement Models

We begin a study of measurement models [7,11,17]. This section only gives an introduction to the theory and we leave more details to later work. If $A \in \mathrm{Ob}(H)$, we define the *Lüders instrument* $\mathcal{L}^A \in \mathrm{In}(H)$ corresponding to A by $\mathcal{L}_x^A(\rho) = A_x^{\frac{1}{2}}\rho A_x^{\frac{1}{2}}$ for all $x \in \Omega_A$, $\rho \in \mathcal{S}(H)$ [18]. Notice that $\mathcal{L}^A$ is a special type of Kraus instrument with Kraus operators $A_x^{\frac{1}{2}}$. Since

$$\mathrm{tr}\left[\mathcal{L}_x^A(\rho)\right] = \mathrm{tr}(A_x^{\frac{1}{2}}\rho A_x^{\frac{1}{2}}) = \mathrm{tr}(\rho A_x)$$

we have that $(\mathcal{L}^A)^\wedge = A$ and every observable is measured by its corresponding Lüders instrument. If A is sharp, then $\mathcal{L}^A$ has the form $\mathcal{L}_x^A(\rho) = A_x\rho A_x$.

A measurement model M is an apparatus that can be employed to gain information about a quantum system S. If S is described by a Hilbert space H, we call H the base space. We interact H with an auxiliary Hilbert space K using an instrument $\mathcal{I} \in \mathrm{In}(H, H \otimes K)$. We then measure a probe observable $P \in \mathrm{Ob}(K)$. The result of this measurement gives information about the state of S or observables on S. We now make this description mathematically precise. A *measurement model* is a four-tuple $M = (H, K, \mathcal{I}, P)$ where H is the *base space* Hilbert space, K is the *auxiliary* Hilbert space, $\mathcal{I} \in \mathrm{In}(H, H \otimes K)$ is the *interaction instrument* and $P \in \mathrm{Ob}(K)$ is the *probe observable*. This definition is a generalization of the measurement models that have already been studied [11, 16]. The *measurement instrument* $\mathcal{M} \in \mathrm{In}(H, H \otimes K)$ for the model M is given by the bi-instrument

$$\mathcal{M}_{xy} = \mathcal{I}_y \circ \mathcal{L}^{I_H \otimes P_x}$$

which results from first applying the interaction and then measuring the probe observable. Thus, for all $\rho \in \mathcal{S}(H)$ we have

$$\mathcal{M}_{xy}(\rho) = \mathcal{L}^{I_H \otimes P_x}\left[\mathcal{I}_y(\rho)\right] = (I_H \otimes P_x)^{\frac{1}{2}}\mathcal{I}_y(\rho)(I_H \otimes P_x)^{\frac{1}{2}}$$

The measurement instrument contains the information obtained from M. In particular, the *marginal measurement instrument* is the instrument $\mathcal{M}^1 \in \mathrm{In}(H, H \otimes K)$ given by

$$\mathcal{M}_x^1(\rho) = \sum_y \mathcal{M}_{xy}(\rho) = \mathcal{L}^{I_H \otimes P_x}\left[\overline{\mathcal{I}}(\rho)\right] = \overline{\mathcal{I}} \circ \mathcal{L}^{I_H \otimes P_x}(\rho)$$

We call the reduced marginal instrument $\mathcal{M}_1^1 \in \mathrm{In}(H)$ the *instrument measured by* M and we obtain

$$\mathcal{M}_{1x}^1(\rho) = \mathrm{tr}_K\left[\mathcal{M}_x^1(\rho)\right] = \mathrm{tr}_K\left[\overline{\mathcal{I}} \circ \mathcal{L}^{I_H \otimes P_x}(\rho)\right]$$

for all $\rho \in \mathcal{S}(H)$. We call the observable $\widehat{\mathcal{M}} \in \mathrm{Ob}(H)$, the *observable measured by* M. Since $\widehat{\mathcal{M}}_1^1$ satisfies

$$\begin{aligned}\mathrm{tr}(\rho\widehat{\mathcal{M}}_{1x}^1) &= \mathrm{tr}\left[\mathcal{M}_{1x}^1(\rho)\right] = \mathrm{tr}\left[\mathcal{L}^{I_H \otimes P_x}(\overline{\mathcal{I}}(\rho))\right]\\ &= \mathrm{tr}\left[(I_H \otimes P_x)^{\frac{1}{2}}\overline{\mathcal{I}}(\rho)(I_H \otimes P_x)^{\frac{1}{2}}\right]\\ &= \mathrm{tr}\left[\overline{\mathcal{I}}(\rho)(I_H \otimes P_x)\right] = \mathrm{tr}\left[\rho\overline{\mathcal{I}}^*(I_H \otimes P_x)\right]\end{aligned}$$

we conclude that $\widehat{\mathcal{M}}_{1x}^1 = \overline{\mathcal{I}}^*(I_H \otimes P_x)$.

Suppose $\mathcal{I}$ is a Holevo instrument $\mathcal{I} = \mathcal{H}^{(A,\alpha)}$, where $A \in \mathrm{Ob}(H)$ and $\alpha = \{\alpha_x : x \in \Omega_A\} \subseteq \mathcal{S}(H \otimes K)$. Then

$$\begin{aligned}\mathcal{M}_{xy}(\rho) &= \mathcal{L}^{I_H \otimes P_x}(\mathcal{I}_y(\rho)) = \mathcal{L}^{I_H \otimes P_x}\left[\mathrm{tr}(\rho A_y)\alpha_y\right]\\ &= \mathrm{tr}(\rho A_y)\mathcal{L}^{I_H \otimes P_x}(\alpha_y)\\ &= \mathrm{tr}(\rho A_x)(I_H \otimes P_x)^{\frac{1}{2}}\alpha_y(I_H \otimes P_x)^{\frac{1}{2}}\end{aligned}$$

and we obtain the instrument measured by M:

$$\mathcal{M}_{1x}^1(\rho) = \sum_y \mathrm{tr}(\rho A_y)\mathrm{tr}_K\left[(I_H \otimes P_x)^{\frac{1}{2}}\alpha_y(I_H \otimes P_x)^{\frac{1}{2}}\right]$$

Since $\mathcal{I}_y^*(a) = \mathrm{tr}(\alpha_y a)A_y$ for all $a \in \mathcal{E}(H \otimes K)$ we have $\overline{\mathcal{I}}^*(a) = \sum_y \mathrm{tr}(\alpha_y a)A_y$. Then the observable measured by M becomes

$$\widehat{\mathcal{M}}_{1x}^1 = \overline{\mathcal{I}}^*(I_H \otimes P_x) = \sum_y \mathrm{tr}\left[\alpha_y(I_H \otimes P_x)\right]A_y$$

which is a post-processing of A because $\sum_x \operatorname{tr}\left[\alpha_y(I_H \otimes P_x)\right] = 1$ for all y. In the particular case where P is sharp and $\alpha_y = \beta_y \otimes \gamma_y$, $\beta_y \in \mathcal{S}(H)$, $\gamma_y \in \mathcal{S}(K)$ we obtain

$$\begin{aligned}\mathcal{M}_{xy}(\rho) &= \operatorname{tr}(\rho A_y)(I_H \otimes P_x)\beta_y \otimes \gamma_y(I_H \otimes P_x)\\ &= \operatorname{tr}(\rho A_y)\beta_y \otimes P_x\gamma_y P_x\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{M}^1_{1x}(\rho) &= \sum_y \operatorname{tr}(\rho A_y)\operatorname{tr}_K(\beta_y \otimes P_x\gamma_y P_x)\\ &= \sum_y \operatorname{tr}(\rho A_y)\operatorname{tr}(P_x\gamma_y)\beta_y\\ &= \sum_y \operatorname{tr}\left[P_x\mathcal{H}^{(A,\gamma)}_y(\rho)\right]\beta_y\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathcal{M}}^1_{1x} &= \sum_y \operatorname{tr}\left[\beta_y \otimes \gamma_y(I_H \otimes P_x)\right]A_y\\ &= \sum_y \operatorname{tr}(\beta_y \otimes \gamma_y P_x)A_x = \sum_y \operatorname{tr}(\gamma_y P_x)A_y\end{aligned}$$

Finally, we introduce the sequential product of measurement models. Let $M = (H, K, \mathcal{I}, P)$ and $M_1 = (H \otimes K, K_1, \mathcal{I}_1, P_1)$ be measurement models where $\mathcal{I} \in \operatorname{In}(H, H \otimes K)$, $P \in \operatorname{Ob}(K)$, $\mathcal{I}_1 \in \operatorname{In}(H \otimes K, H \otimes K \otimes K_1)$, $P_1 \in \operatorname{Ob}(K_1)$. The *sequential product of M then M_1* is the measurement model

$$M_2 = M \circ M_1 = (H, K \otimes K_1, \mathcal{I}_2, P_2)$$

where $\mathcal{I}_2 \in \operatorname{In}(H, H \otimes K \otimes K_1)$ is given by $\mathcal{I}_2 = \mathcal{I} \circ \mathcal{I}_1$ and $P_2 \in \operatorname{Ob}(K \otimes K_1)$ is given by $P_{2xy} = P_x \otimes P_{1y}$. The corresponding measurement instrument for M_2 because the 4-instrument $\mathcal{M} \in \operatorname{In}(H, H \otimes K \otimes K_1)$ defined as

$$\mathcal{M}_{xyx'y'} = \mathcal{I}_{2x'y'} \circ \mathcal{L}^{I_H \otimes P_{2xy}}$$

Hence,

$$\begin{aligned}\mathcal{M}_{xyx'y'}(\rho) &= \mathcal{L}^{I_H \otimes P_{2xy}}\left[\mathcal{I}_{2x'y'}(\rho)\right]\\ &= (I_H \otimes P_{2xy})^{\frac{1}{2}}\mathcal{I}_{2x'y'}(\rho)(I_H \otimes P_{2xy})^{\frac{1}{2}}\\ &= (I_H \otimes P_x \otimes P_{1y})^{\frac{1}{2}}\mathcal{I}_{1x'}(\mathcal{I}_{y'}(\rho))(I_H \otimes P_x \otimes P_{1y})^{\frac{1}{2}}\end{aligned}$$

The marginal measurement $\mathcal{M}^1_{xy} \in \operatorname{In}(H, H \otimes K \otimes K_1)$ becomes

$$\begin{aligned}\mathcal{M}^1_{xy}(\rho) &= \sum_{x',y'} \mathcal{M}_{xyx'y'}\\ &= (I_H \otimes P_x \otimes P_{1y})^{\frac{1}{2}}\overline{\mathcal{I}}_1(\overline{\mathcal{I}}(\rho))(I_H \otimes P_x \otimes P_{1y})^{\frac{1}{2}}\end{aligned}$$

We then obtain the instrument $\mathcal{M}^1_{1xy} \in \operatorname{In}(H)$ measured by M_2 as

$$\mathcal{M}^1_{1xy}(\rho) = \operatorname{tr}_{K \otimes K_1}\left[\mathcal{M}^1_{xy}(\rho)\right]$$

and the observable $\widehat{\mathcal{M}}^1_{1xy}$ measured by M_2 becomes

$$\widehat{\mathcal{M}}^1_{1xy} = \overline{\mathcal{I}}^*\left[\overline{\mathcal{I}}^*_1(I_H \otimes P_x \otimes P_{1y})\right]$$

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