Dual Instruments and Sequential Products of Observables

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We first show that every operation possesses an unique dual operation and measures an unique effect. If a and b are effects and J is an operation that measures a, we define the sequential product of a then b relative to J. Properties of the sequential product are derived and are illustrated in terms of Lüders and Holevo operations. We next extend this work to the theory of instruments and observables. We also define the concept of an instrument (observable) conditioned by another instrument (observable). Identity, state-constant and repeatable instruments are considered. Sequential products of finite observables relative to Lüders and Holevo instruments are studied.

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1 Sequential Products of Effects

Let S be a quantum system described by a complex Hilbert space H. One of the main points of this article is that the sequential product of two observables for S depends on the instrument $\mathcal{I}$ employed to measure the first observable and is independent of the instrument used to measure the second. In this way, the measurement of the first observable influences the measurement of the second but not vice versa. As we shall see, the sequential product is defined in terms of the dual $\mathcal{I}^*$ of $\mathcal{I}$.

We denote the set of bounded linear operators on H by $\mathcal{L}(H)$ and the set of trace-class operators on H by $\mathcal{T}(H)$. For $A, B \in \mathcal{L}(H)$ we write $A \leq B$ if $\langle\phi, A\phi\rangle \leq \langle\phi, B\phi\rangle$ for all $\phi \in H$. We say that $A \in \mathcal{L}(H)$ is *positive* if $A \geq 0$ and A is an *effect* if $0 \leq A \leq I$ where $0, I$ are the zero and identity operators on H, respectively [1–5]. The set of effects on H is denoted by $\mathcal{E}(H)$. We interpret effects as measurements that have two possible outcomes, true and false. If $a \in \mathcal{E}(H)$, then its *complement* $a' = I - a$ is true if and only if a is false. If $a, b \in \mathcal{E}(H)$ and $a + b \in \mathcal{E}(H)$ we write $a \perp b$ and interpret $a + b$ as the statistical sum of the measurements a and b. Of course, $0 \perp a$ for all $a \in \mathcal{E}(H)$ and we interpret 0 as the effect that is always false. Similarly, $1 \perp a$ if and only if $a = 0$ and 1 is the effect that is always true. Moreover, $b \perp a$ if and only if $b \leq a'$. A map $K\colon \mathcal{E}(H) \to \mathcal{E}(H)$ is *additive* if $K(a) \perp K(b)$ whenever $a \perp b$ and we have that $K(a+b) = K(a) + K(b)$. If K is additive, then K preserves order because if $a \leq b$, then there exists a $c \in \mathcal{E}(H)$ such that $a + c = b$ and we obtain

$$K(a) \leq K(a) + K(c) = K(a + c) = K(b)$$

If K is additive and $K(I) = I$, then K is a *morphism* [1, 6–8].

A *state* for S is a positive operator $\rho \in \mathcal{T}(H)$ such that $\mathrm{tr}\,(\rho) = 1$. We denote the set of states by $\mathcal{S}(H)$ and interpret $\rho \in \mathcal{S}(H)$ as an initial condition for the system S [1, 6]. We define the *probability that* $a \in \mathcal{E}(H)$

is true when S is in the state ρ by $\mathcal{P}_\rho(a) = \mathrm{tr}\,(\rho a)$. It follows that $\mathcal{P}_\rho(a') = 1 - \mathcal{P}_\rho(a)$ and $a \le b$ if and only if $\mathcal{P}_\rho(a) \le \mathcal{P}_\rho(b)$ for all $\rho \in \mathcal{S}(H)$. An *operation* on H is a completely positive linear map $J\colon \mathcal{L}(H) \to \mathcal{L}(H)$ that is trace non-increasing for $\mathcal{T}(H)$ operators. We denote the set of operations on H by $\mathcal{O}(H)$. It can be shown [1,2,6,9] that every $J \in \mathcal{O}(H)$ has a *Kraus decomposition* $J(A) = \sum C_i A C_i^*$, $A \in \mathcal{L}(H)$, where $C_i \in \mathcal{L}(H)$ satisfy $\sum C_i^* C_i \le I$. This condition follows from the fact that for every $A \in \mathcal{T}(H)$ with $A \ge 0$ we have that

$$\begin{aligned}\mathrm{tr}\left(\sum C_i^* C_i A\right) &= \sum \mathrm{tr}\,(C_i^* C_i A) = \sum \mathrm{tr}\,(C_i A C_i^*)\\ &= \mathrm{tr}\left(\sum C_i A C_i^*\right) \le \mathrm{tr}\,(A)\end{aligned}$$

holds if and only if $\sum C_i^* C_i \le I$. If an operation preserves the trace, it is called a *channel* [1, 5, 6, 10]. A *dual operation* on H is a completely positive linear map $K\colon \mathcal{L}(H) \to \mathcal{L}(H)$ that satisfies $K\colon \mathcal{E}(H) \to \mathcal{E}(H)$. It follows that $K|_{\mathcal{E}(H)}$ is additive. We denote the set of dual operations on H by $\mathcal{O}^*(H)$.

Theorem 1. If $J\colon \mathcal{L}(H) \to \mathcal{L}(H)$ is an operation, then there exists a unique $J^* \in \mathcal{O}^*(H)$ such that $\mathrm{tr}\,[\rho J^*(a)] = \mathrm{tr}\,[J(\rho)a]$ for all $a \in \mathcal{E}(H)$, $\rho \in \mathcal{S}(H)$. Conversely, if $K \in \mathcal{O}^*(H)$, then there exists a unique $J \in \mathcal{O}(H)$ such that $J^* = K$. Moreover, J is a channel if and only if $J^*(I) = I$.

Proof. Let $J \in \mathcal{O}(H)$ with Kraus decomposition $J(A) = \sum C_i A C_i^*$ where $\sum C_i^* C_i \le I$ and define $J^*(A) = \sum C_i^* A C_i$ for all $A \in \mathcal{L}(H)$. If $a \in \mathcal{E}(H)$ and $\phi \in H$, since $0 \le a \le I$, we have that

$$\begin{aligned}\langle \phi, J^*(a)\phi\rangle &= \left\langle \phi, \sum C_i^* a C_i \phi\right\rangle\\ &= \sum \langle C_i\phi, a C_i\phi\rangle \le \sum \langle C_i\phi, C_i\phi\rangle\\ &= \left\langle \phi, \sum C_i^* C_i \phi\right\rangle \le \langle \phi, \phi\rangle\end{aligned}$$

Moreover, $\langle \phi, J^*(a)\phi\rangle \ge 0$ so $0 \le J^*(a) \le I$ and we conclude that $J^*(a) \in \mathcal{E}(H)$. Since J^* also has a Kraus decomposition, it follows that $J^* \in \mathcal{O}^*(H)$. The duality condition holds because

$$\begin{aligned}\mathrm{tr}\,[\rho J^*(a)] &= \mathrm{tr}\left(\rho \sum C_i^* a C_i\right) = \sum \mathrm{tr}\,(\rho C_i^* a C_i)\\ &= \sum \mathrm{tr}\,(C_i \rho C_i^* a) = \mathrm{tr}\left[\sum C_i \rho C_i^* a\right]\\ &= \mathrm{tr}\,[J(\rho)a] \qquad (1)\end{aligned}$$

for all $a \in \mathcal{E}(H)$, $\rho \in \mathcal{S}(H)$. To show that J^* is unique, suppose $K \in \mathcal{O}^*(H)$ satisfies $\mathrm{tr}\,[\rho K(a)] = \mathrm{tr}\,[J(\rho)a]$ for all $a \in \mathcal{E}(H)$, $\rho \in \mathcal{S}(H)$. Then $\mathrm{tr}\,[\rho K(a)] = \mathrm{tr}\,[\rho J^*(a)]$ for all $a \in \mathcal{E}(H)$, $\rho \in \mathcal{S}(H)$ so $K = J^*$. Conversely, let $K \in \mathcal{O}^*(H)$ with Kraus decomposition $K(a) = \sum C_i^* a C_i$. Since $K\colon \mathcal{E}(H) \to \mathcal{E}(H)$ and $I \in \mathcal{E}(H)$ we have that $K(I) \le I$. Hence,

$$\sum C_i^* C_i = K(I) \le I$$

It follows that the map $J(A) = \sum C_i A C_i^*$ is an operation. As in (1) we have that

$$\mathrm{tr}\,[\rho K(a)] = \mathrm{tr}\,[J(\rho)a] = \mathrm{tr}\,[\rho J^*(a)]$$

We conclude that $J^* = K$ and as before, J is unique. If $J^*(I) = I$, then

$$\mathrm{tr}\,[J(\rho)] = \mathrm{tr}\,[J(\rho)I] = \mathrm{tr}\,[\rho J^*(I)] = \mathrm{tr}\,(\rho) = 1$$

for every $\rho \in \mathcal{S}(H)$ so J is a channel. Conversely, if J is a channel, then

$$\mathrm{tr}\,[\rho J^*(I)] = \mathrm{tr}\,[J(\rho)I] = \mathrm{tr}\,[J(\rho)] = 1$$

so $J^*(I) = I$. □

In the proof of Theorem 1, we defined $J^*(A) = \sum C_i^* A C_i$, where J has Kraus decomposition $J(A) = \sum C_i A C_i^*$. The Kraus operators C_i are not unique and there can be many such operators [2, 5]. Suppose we have another Kraus decomposition $J(A) = \sum D_j A D_j^*$. By uniqueness, we conclude that $J^*(A) = \sum D_j^* A D_j$ so the form of the Kraus operators is immaterial. We say that an operation J *measures* an effect a if

$$\mathrm{tr}\,[J(\rho)] = \mathrm{tr}\,(\rho a) = \mathcal{P}_\rho(a)$$

for every $\rho \in \mathcal{S}(H)$. We think of J as an apparatus that can be employed to measure the effect a [10–13]. Then $\mathrm{tr}\,[J(\rho)]$ gives the probability that a is true when the system S is in the state ρ. The operation J gives more information than the effect a. If $\alpha \in \mathcal{T}(H)$ with $\alpha > 0$ we define its corresponding state to be $\widetilde{\alpha} = \alpha/\mathrm{tr}\,(\alpha)$. After an operation J is performed, the state ρ is *updated* to the state $(J\rho)^\sim$ [10–13].

If $0 \le \lambda_i \le 1$ with $\sum \lambda_i = 1$ and $a_i \in \mathcal{E}(H)$, then it is clear that $\sum \lambda_i a_i \in \mathcal{E}(H)$ and if $\rho_i \in \mathcal{S}(H)$ we have that $\sum \lambda_i \rho_i \in \mathcal{S}(H)$. We conclude that $\mathcal{E}(H)$ and $\mathcal{S}(H)$ are closed under convex combinations and hence form convex sets. In a similar way, if $J_i \in \mathcal{O}(H)$, $K_i \in \mathcal{I}^*(H)$, then $\sum \lambda_i J_i \in \mathcal{O}(H)$ and $\sum \lambda_i K_i \in \mathcal{O}^*(H)$ so $\mathcal{O}(H)$ and $\mathcal{O}^*(H)$ form convex sets. If $J_1, J_2 \in \mathcal{O}(H)$, we define their *sequential product* $J_1 \circ J_2 \in \mathcal{O}(H)$ by $J_1 \circ J_2(A) = J_2\,(J_1(A))$ [10–13]. Physically, $J_1 \circ J_2$ specifies the operation obtained by first employing the operation J_1 and then employing J_2. In a similar way, if $K_1, K_2 \in \mathcal{O}^*(H)$, their *sequential product* $K_1 \circ K_2 \in \mathcal{O}^*(H)$ is $K_1 \circ K_2(A) = K_2\,(K_1(A))$.

Theorem 2. (i) An operation J measures an unique effect given by $\widehat{J} = J^*(I)$. (ii) If $0 \le \lambda_i \le 1$ with $\sum \lambda_i = 1$ and $J_i \in \mathcal{O}(H)$, then $(\sum \lambda_i J_i)^* = \sum \lambda_i J_i^*$ and $(\sum \lambda_i J_i)^\wedge = \sum \lambda_i \widehat{J_i}$. (iii) If we also have $0 \le \mu_j \le 1$ with $\sum \mu_j = 1$ and $K_j \in \mathcal{O}(H)$ then

$$\left(\sum_i \lambda J_i\right) \circ \left(\sum_j \mu_j K_j\right) = \sum_{i,j} \lambda_i \mu_j J_i \circ K_j$$

and this result also holds if $J_i, K_j \in \mathcal{O}^*(H)$. (iv) If $J, K \in \mathcal{O}(H)$ then $(J \circ K)^* = K^* \circ J^*$ and $(J \circ K)^\wedge = J^*(\widehat{K})$. (v) The following statements are equivalent: (a) J is a channel, (b) $\widehat{J} = I$, (c) $J^*(I) = I$, (d) $(K \circ J)^\wedge = \widehat{K}$ for all $K \in \mathcal{O}(H)$.

Proof. (i) Since

$$\mathrm{tr}\,[J(\rho)] = \mathrm{tr}\,[J(\rho)I] = \mathrm{tr}\,[\rho J^*(I)]$$

for all $\rho \in \mathcal{S}(H)$, we conclude that J measures $J^*(I)$. For uniqueness, if J also measures a, then

$$\mathrm{tr}\,(\rho a) = \mathrm{tr}\,[J(\rho)] = \mathrm{tr}\,[\rho J^*(I)]$$

for all $\rho \in \mathcal{S}(H)$ so $a = J^*(I)$.
(ii) Since

$$\begin{aligned}\mathrm{tr}\left[\rho\left(\sum \lambda_i J_i\right)^*(a)\right] &= \mathrm{tr}\left[\left(\sum \lambda_i J_i\right)(\rho)a\right]\\ &= \mathrm{tr}\left[\sum \lambda_i J_i(\rho)a\right]\\ &= \sum \lambda_i \mathrm{tr}\,[J_i(\rho)a]\\ &= \sum \lambda_i \mathrm{tr}\left[\rho J_i^*(a)\right]\\ &= \mathrm{tr}\left[\rho \sum \lambda_i J_i^*(a)\right]\end{aligned}$$

for all $\rho \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$, it follows that $(\sum \lambda_i J_i)^* = \sum \lambda_i J_i^*$. We then obtain

$$\left(\sum \lambda_i J_i\right)^\wedge = \left(\sum \lambda_i J_i\right)^*(I) = \sum \lambda_i J_i^*(I) = \sum \lambda_i \widehat{J_i}$$

which gives the result.
(iii) For all $A \in \mathcal{L}(H)$ we obtain

$$\begin{aligned}\left(\sum_i \lambda_i J_i\right) \circ \left(\sum_j \mu_j K_j\right)(A) &= \sum_j \mu_j K_j\left(\sum_i \lambda_i J_i(A)\right)\\ &= \sum_{i,j} \lambda_i \mu_j K_j\,(J_i(A))\\ &= \sum_{i,j} \lambda_i \mu_j J_i \circ K_j(A)\end{aligned}$$

and the result follows. The proof for $J_i, K_j \in \mathcal{O}^*(H)$ is similar.
(iv) For all $\rho \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$ we have that

$$\begin{aligned}\mathrm{tr}\,[\rho(J \circ K^*)(a)] &= \mathrm{tr}\,[(J \circ K)(\rho)a] = \mathrm{tr}\,[K\,(J(\rho))\,a]\\ &= \mathrm{tr}\,[J(\rho)K^*(a)] = \mathrm{tr}\,[\rho J^*\,(K^*(a))]\\ &= \mathrm{tr}\,[\rho(K^* \circ J^*)(a)]\end{aligned}$$

Hence, $(J \circ K)^* = K^* \circ J^*$. It then follows from (i) that

$$(J \circ K)^\wedge = (J \circ K)^*(I) = (K^* \circ J^*)(I) = J^*\,(K^*(I)) = J^*(\widehat{K})$$

(v) (a)$\Rightarrow$(b) If J is a channel, then for every $\rho \in \mathcal{S}(H)$ we have that

$$\mathrm{tr}\,(\rho I) = 1 = \mathrm{tr}\,[J(\rho)I] = \mathrm{tr}\,[\rho J^*(I)] = \mathrm{tr}\,(\rho \widehat{J})$$

Hence, $\widehat{J} = I$. (b)$\Leftrightarrow$(c) This follows from (i). (c)$\Rightarrow$(d) If $\widehat{J} = J^*(I) = I$, applying (i) and (iv) gives

$$\begin{aligned}(K \circ J)^\wedge &= (K \circ J)^*(I) = (J^* \circ K^*)(I) = K^*\,(J^*(I))\\ &= K^*(\widehat{J}) = K^*(I) = \widehat{K}\end{aligned}$$

(d)$\Rightarrow$(a) Suppose (d) holds and let K be the identity channel $K(\rho) = \rho$ for all $\rho \in \mathcal{S}(H)$. Then $\widehat{K} = I$ so by (d) we have that

$$\widehat{J} = (K \circ J)^\wedge = \widehat{K} = I$$

We then obtain for all $\rho \in \mathcal{S}(H)$ that

$$\mathrm{tr}\,[J(\rho)] = \mathrm{tr}\,[J(\rho)I] = \mathrm{tr}\,[\rho J^*(I)] = \mathrm{tr}\,(\rho \widehat{J}) = \mathrm{tr}\,(\rho) = 1$$

Hence, J is a channel. □

The proof of the following result is similar to Theorem 2(ii)

Corollary 3. If $J, K \in \mathcal{O}(H)$ and $J + K \in \mathcal{O}(H)$, then $(J + K)^* = J^* + K^*$ and $(J + K)^\wedge = \widehat{J} + \widehat{K}$.

If $a, b \in \mathcal{E}(H)$ and $J \in \mathcal{O}(H)$ measures a so that $\widehat{J} = a$, we define the *sequential product of a then b relative to J* by $a\,[J]\,b = J^*(b)$. We interpret $a\,[J]\,b$ as the effect that results from first measuring a with the operation J and then measuring b. In this way, the measurement of a can influence (or interfere with) b, but since we measure b second, the measurement of b does not influence a. An important point is that $a\,[J]\,b$ depends on J. As we shall see, there are many operations that measure a so if $\widehat{K} = a$, then $a\,[J]\,b \neq a\,[K]\,b$, in general. Moreover, $a\,[J]\,b$ does not depend on an operation that measures b.

Theorem 4. (i) If $\widehat{J} = a$, $\widehat{K} = b$, then $a\,[J]\,b = (J \circ K)^\wedge$. (ii) $a\,[J]\,b \leq a$ for all $a, b \in \mathcal{E}(H)$. (iii) If $0 \leq \lambda \leq 1$ and $\widehat{J} = a$, then $(\lambda a)\,[\lambda J]\,b = \lambda\,(a\,[J]\,b) = a\,[J]\,(\lambda b)$. (iv) $a\,[J]\,I = a$ for all $a \in \mathcal{E}(H)$. (v) $a\,[J]\,b' = a - J^*(a)$. (vi) If $\widehat{J_i} = a_i$, $0 \leq \lambda_i \leq 1$ with $\sum \lambda_i = 1$ and $0 \leq \mu_j \leq 1$ with $\sum \mu_j = 1$, then for any $b_i \in \mathcal{E}(H)$ we have that

$$\left(\sum \lambda_i a_i\right)\left[\sum \lambda_i J_i\right]\left(\sum \mu_j b_j\right) = \sum_{i,j} \lambda_i \mu_j a_i\,[J_i]\,b_j$$

Proof. (i) By Theorem 2(iv) we obtain

$$a\,[J]\,b = J^*(b) = J^*(\widehat{K}) = (J \circ K)^\wedge$$

(ii) This follows from

$$a\,[J]\,b = J^*(b) \leq J^*(I) = \widehat{J} = a$$

(iii) We have that

$$(\lambda a)\,[\lambda J]\,b = (\lambda J)^*(b) = \lambda J^*(b) = \lambda\,(a\,[J]\,b)$$

and

$$\lambda J^*(b) = J^*(\lambda b) = a\,[J]\,(\lambda b)$$

(iv) This follows from

$$a\,[J]\,I = J^*(I) = \widehat{J} = a$$

(v) We have that

$$\begin{aligned} a\,[J]\,a' &= a\,[J]\,(I-a) = J^*(I-a) = J^*(I) - J^*(a) \\ &= \widehat{J} - \widehat{J}(a) = a - J^*(a) \end{aligned}$$

(vi) Applying Theorem 2(ii) we obtain

$$\begin{aligned} \left(\sum \lambda_i a_i\right)\left[\sum \lambda_i J_i\right]\left(\sum \mu_j b_j\right) &= \left(\sum \lambda_i J_i\right)^*\left(\sum \mu_j b_j\right) \\ &= \sum \lambda_j J_i^*\left(\sum \mu_j b_j\right) \\ &= \sum_{i,j} \lambda_i \mu_j J_i^*(b_j) \\ &= \sum_{i,j} \lambda_i \mu_j a_i\,[J_i]\,b_j \end{aligned}$$

□

We end this section with some definitions suggested by the theory. We say that $a, b \in \mathcal{E}(H)$ *commute relative* to a subset $\mathcal{R} \subseteq O(H)$ if there exist operations $J, K \in \mathcal{R}$ such that

$$a\,[J]\,b = b\,[K]\,a \tag{2}$$

Of course, (2) is equivalent to $J^*(b) = K^*(a)$, where $\widehat{J} = a$ and $\widehat{K} = b$. In particular, when (2) holds, then a and b commute relative to $\{J, K\} \subseteq O(H)$. When $\mathcal{R} = O(H)$, we just say that a and b *commute*. Since

$$a\,[J]\,0 = J^*(0) = 0 = 0^*(a) = 0\,[0]\,a$$

we conclude that any $a \in \mathcal{E}(H)$ commutes with 0. Similarly,

$$a\,[J]\,I = J^*(I) = a = I^*(a) = I\,[I]\,a$$

So any $a \in \mathcal{E}(H)$ commutes with I. Suppose a commutes with b so (2) holds. If $0 \le \lambda \le 1$, then

$$a\,[J]\,(\lambda b) = \lambda b\,[\lambda K]\,a$$

Hence, a commutes with λb. We do not know if the following conjecture holds.

Conjecture 1. If a commutes with b and c where $b \perp c$, then a commutes with $b + c$.

Let $a, b \in \mathcal{E}(H)$ and let $J, K \in O(H)$ with $\widehat{J} = a$, $\widehat{K} = a'$. We define the effect b *conditioned* by the effect a *relative* to $\{J, K\}$ by

$$(b\,|J, K|\,a) = a\,[J]\,b + a'\,[K]\,b$$

We interpret $(b\,|J, K|\,a)$ as the effect b conditioned on whether a is true or false as measured by the operations, J, K respectively, In terms of probabilities, we have

$$\begin{aligned} \mathcal{P}_\rho\,(b\,|J, K|\,a) &= \mathrm{tr}\,[\rho J^*(b)] + \mathrm{tr}\,[\rho K^*(b)] \\ &= \mathrm{tr}\,[J(\rho)b] + \mathrm{tr}\,[K(\rho)b] \\ &= \mathcal{P}_\rho(a)\mathcal{P}_{\widetilde{J(\rho)}}(b) + \mathcal{P}_\rho(a')\mathcal{P}_{\widetilde{K(\rho)}}(b) \end{aligned} \tag{3}$$

Equation (3) is a type of Bayes' rule where $\mathcal{P}_{\widetilde{J(\rho)}}$ is the probability that b is true given that a is true and $\mathcal{P}_{\widetilde{K(\rho)}}(b)$ is the probability that b is true given that a is false. We say that b *is not influenced by* a relative to $\{J, K\}$ if $b = (b\,|J, K|\,a)$.

2 Lüders and Holevo Operations

The most important example of an operation is the Lüders operation L^a, $a \in \mathcal{E}(H)$, given by $L^a(A) = a^{\frac{1}{2}} A a^{\frac{1}{2}}$. Since

$$\mathrm{tr}\,[L^a(\rho)b] = \mathrm{tr}\,(a^{\frac{1}{2}}\rho a^{\frac{1}{2}} b) = \mathrm{tr}\,(\rho a^{\frac{1}{2}} b a^{\frac{1}{2}}) = \mathrm{tr}\,[\rho (L^a)^*(b)]$$

we have that $(L^a)^*(b) = a^{\frac{1}{2}} b a^{\frac{1}{2}} = L^a(b)$ so L^a is *self-adjoint* in the sense that $L^a = (L^a)^*$. Moreover, $(L^a)^\wedge = (L^a)^*(I) = a$ so L^a measures a. In fact, L^a is the unique Lüders operation that measures a. An effect a is *sharp* if a is a projection. We denote the set of Lüders operations by $\mathcal{L}$.

Theorem 5. (i) $(L^a \circ J)^\wedge = a^{\frac{1}{2}} \widehat{J} a^{\frac{1}{2}}$ for all $J \in O(H)$. (ii) $a\,[L^a]\,b = a^{\frac{1}{2}} b a^{\frac{1}{2}} = L^a(b)$. (iii) $(J \circ L^a)^\wedge = J^*(a)$. (iv) $(L^a \circ L^b)^\wedge = a^{\frac{1}{2}} b a^{\frac{1}{2}}$. (v) a commutes with b relative to $\mathcal{L}$ if and only if $ab = ba$, that is, a and b commute in the usual operator sense. (vi) If a is sharp, then b is not influenced by a relative to $\left\{L^a, L^{a'}\right\}$ if and only if $ab = ba$.

Proof. (i) By Theorem 2(iv) we have that

$$(L^a \circ J)^\wedge = (L^a)^*(\widehat{J}) = a^{\frac{1}{2}} \widehat{J} a^{\frac{1}{2}}$$

(ii) This follows from

$$a\,[L^a]\,b = (L^a)^*(b) = a^{\frac{1}{2}} b a^{\frac{1}{2}} = L^a(b)$$

(iii) Applying Theorem 2(iv) we obtain

$$(J \circ L^a)^\wedge = J^*\left[(L^a)^\wedge\right] = J^*(a)$$

(iv) follows from (i).
(v) We have that a commutes with b relative to $\mathcal{L}$ if and only if

$$a^{\frac{1}{2}} b a^{\frac{1}{2}} = a\,[L^a]\,b = b\left[L^b\right]a = b^{\frac{1}{2}} a b^{\frac{1}{2}}$$

which holds if and only if $ab = ba$ [8]. (vi) If a is sharp then $a^{\frac{1}{2}} = a$ so b is not influenced by a relative to $\left\{L^a, L^{a'}\right\}$ if and only if

$$b = a\,[L^a]\,b + a'\left[L^{a'}\right]b = aba + a'ba'$$

Multiplying on left by a gives $ab = aba$. Hence, $ab = (ab)^* = b^*a^* = ba$. Conversely, if $ab = ba$, then

$$aba + a'ba' = ab + a'b = b \qquad \square$$

Theorem 5(i) and (iii) show that $L^a \circ J$ measures $a^{\frac{1}{2}}\widehat{J}a^{\frac{1}{2}}$ and $J \circ L^a$ measures $J^*(a)$ for all $J \in \mathcal{O}(H)$. We call $a \square b = a\left[L^a\right]b = a^{\frac{1}{2}}ba^{\frac{1}{2}}$ the *standard sequential product* of a and b [7, 8, 10, 13]. Of course, if $J \neq L^a$ then $a\left[J\right]b \neq a^{\frac{1}{2}}ba^{\frac{1}{2}}$, in general. Theorem 4(vi) shows that in a certain sense, a sequential product preserves convex combinations. This does not imply that when $0 \leq \lambda \leq 1$ we have

$$\left[\lambda a + (1-\lambda)b\right] \square c = \lambda a \square c + (1-\lambda) b \square c$$

which does not hold in general. In fact, we have that

$$\left[\lambda a + (1-\lambda)b\right] \square c = \left[\lambda a + (1-\lambda)b\right]^{\frac{1}{2}} c \left[\lambda a + (1-\lambda)b\right]^{\frac{1}{2}}$$

On the other hand

$$\lambda a \square c + (1-\lambda) b \square c = \lambda a^{\frac{1}{2}} c a^{\frac{1}{2}} + (1-\lambda) b^{\frac{1}{2}} c b^{\frac{1}{2}}$$

For $\alpha \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$, we call $H_{(\alpha,a)}(\rho) = \mathrm{tr}\,(\rho a)\alpha$ the *Holevo operation with state α and effect a* [14]. The next theorem shows that the sequential product of any operation with a Holevo operation is again a Holevo operation. It also shows that $\widehat{H}_{(\alpha,a)} = a$ for any $\alpha \in \mathcal{S}(H)$. This illustrates the fact that an effect can be measured by many operations. We denote the set of Holevo operations by $\mathcal{H}$.

Theorem 6. (i) $H^*_{(\alpha,a)}(b) = \mathrm{tr}\,(\alpha b)a$ and $\widehat{H}_{(\alpha,a)} = a$ for all $\alpha \in \mathcal{S}(H)$, $a, b \in \mathcal{E}(H)$. (ii) $H_{(\alpha,a)} \circ J = H_{(\widetilde{J\alpha},\mathrm{tr}\,(J\alpha)a)}$ and $J \circ H_{(\alpha,a)} = H_{(\alpha,J^*(a))}$. (iii) $(H_{(\alpha,a)} \circ J)^\wedge = \mathrm{tr}\,\left[J(\alpha)\right]a$ and $(J \circ H_{(\alpha,a)})^\wedge = J^*(a)$. (iv) $H_{(\beta,b)} \circ H_{(\alpha,a)} = H_{(\alpha,\mathrm{tr}\,(\beta a)b)}$. (v) $a\left[H_{(\alpha,a)}\right]b = \left[\mathrm{tr}\,(\alpha b)\right]a$. (vi) a commutes with b relative to $\mathcal{H}$ if and only if there exists $\alpha, \beta \in \mathcal{S}(H)$ such that $\mathrm{tr}\,(\alpha b)a = \mathrm{tr}\,(\beta a)b$. (vii) a does not influence b relative to $\left\{H_{(\alpha,a)}, H_{(\beta,a')}\right\}$ if and only if $b = \mathrm{tr}\,\left[(\alpha-\beta)b\right]a + \mathrm{tr}\,(\beta b)I$. In particular, if $\alpha = \beta$ then $b = \mathrm{tr}\,(\alpha b)I$. (viii) $\left(b \left| H_{(\alpha,a)}, H_{(\beta,a')} \right| a\right) = \mathrm{tr}\,\left[(\alpha-\beta)b\right]a + \mathrm{tr}\,(\beta b)I$.

Proof. (i) We have that

$$\begin{aligned}\mathrm{tr}\,\left[\rho H^*_{(\alpha,a)}b\right] &= \mathrm{tr}\,\left[H_{(\alpha,a)}(\rho)b\right] = \mathrm{tr}\,\left[\mathrm{tr}\,(\rho a)\alpha b\right] \\ &= \mathrm{tr}\,(\rho a)\mathrm{tr}\,(\alpha b) = \mathrm{tr}\,\left[\rho\,\mathrm{tr}\,(\alpha b)a\right]\end{aligned}$$

It follows that $H^*_{(\alpha,a)}(b) = \mathrm{tr}\,(\alpha b)a$. We conclude that $H_{(\alpha,a)}$ measures the effect

$$\widehat{H}_{(\alpha,a)} = H^*_{(\alpha,a)}(I) = \mathrm{tr}\,(\alpha I)a = a$$

(ii) For all $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}(H_{(\alpha,a)} \circ J)(\rho) &= J\left[H_{(\alpha,a)}(\rho)\right] = J\left[\mathrm{tr}\,(\rho a)\alpha\right] \\ &= \mathrm{tr}\,(\rho a)J(\alpha) \\ &= \mathrm{tr}\,\left[\rho\,\mathrm{tr}\,(J\alpha)a\right]\widetilde{J\alpha} = H_{(\widetilde{J\alpha},\mathrm{tr}\,(J\alpha)a)}(\rho)\end{aligned}$$

and the result follows. Moreover, for all $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}(J \circ H_{(\alpha,a)})(\rho) &= H_{(\alpha,a)}\left[J(\rho)\right] = \mathrm{tr}\,\left[J(\rho)a\right]\alpha \\ &= \mathrm{tr}\,\left[\rho J^*(a)\right]\alpha = H_{(\alpha,J^*(a))}(\rho)\end{aligned}$$

and the result follows.
(iii) These follow from (i) and (ii).
(iv) Applying (i) and (ii) gives

$$H_{(\beta,b)} \circ H_{(\alpha,a)} = H_{\left(\alpha,H^*_{(\beta,b)}\right)}(a) = H_{(\alpha,\mathrm{tr}\,(\beta a)b)}$$

(v) Applying (i) gives

$$a\left[H_{(\alpha,a)}\right]b = H^*_{(\alpha,a)}(b) = \mathrm{tr}\,(\alpha b)a$$

(vi) By (v) we have that $a\left[H_{(\alpha,a)}\right]b = \mathrm{tr}\,(\alpha b)a$ and $b\left[H_{(\beta,b)}\right]a = \mathrm{tr}\,(\beta a)b$. Hence, $a\left[H_{(\alpha,a)}\right]b = b\left[H_{(\beta,b)}\right]a$ if and only if $\mathrm{tr}\,(\alpha b)a = \mathrm{tr}\,(\beta a)b$.
(vii) For all $\alpha, \beta \in \mathcal{S}(H)$ we have by (v) that

$$\begin{aligned}a\left[H_{(\alpha,a)}\right]b + a'\left[H_{(\beta,a')}\right]b &= H^*_{(\alpha,a)}(b) + H^*_{(\beta,a')}(b) \\ &= \mathrm{tr}\,(\alpha b)a + \mathrm{tr}\,(\beta b)a' \\ &= \mathrm{tr}\,(\alpha b)a + \mathrm{tr}\,(\beta b)I - \mathrm{tr}\,(\beta b)a \\ &= \mathrm{tr}\,\left[(\alpha-\beta)b\right]a + \mathrm{tr}\,(\beta b)I\end{aligned}$$

The result follows.
(viii) This follows from (vii). $\square$

Theorem 4(iv) shows that $a\left[J\right]I = a$ for all $a \in \mathcal{E}(H)$. We can use Holevo operations to show that $I\left[J\right]a \neq a$, in general. Applying Theorem 6(i) we have that

$$I\left[H_{(\alpha,I)}\right]a = H^*_{(\alpha,I)}(a) = \mathrm{tr}\,(\alpha a)I \neq a$$

in general.

3 Instruments and Observables

We now extend our previous work to the theory of instruments and observables. If $(\Omega_\mathcal{I}, \mathcal{F}_\mathcal{I})$ is a measurable space, an *instrument* on H with *outcome space* $(\Omega_\mathcal{I}, \mathcal{F}_\mathcal{I})$ is an operation-valued measure on $\mathcal{F}_\mathcal{I}$. That is, $\Delta \mapsto \mathcal{I}(\Delta) \in \mathcal{O}(H)$ is countably additive relative to a suitable topology and $\mathcal{I}(\Omega_\mathcal{I}) = \overline{\mathcal{I}}$ is a channel [1, 2, 5, 6]. We denote the set of instruments on H by $\mathcal{I}n(H)$. We interpret an instrument as an apparatus that can be employed to perform measurements. Then $\mathcal{I}(\Delta)$ is the operation that results when a measurement of $\mathcal{I}$ gives an outcome in Δ. For any $\rho \in \mathcal{S}(H)$, we call $\Phi^\mathcal{I}_\rho(\Delta) = \mathrm{tr}\,\left[\mathcal{I}(\Delta)(\rho)\right]$ the *distribution* of $\mathcal{I}$ in the state ρ and interpret $\Phi^\mathcal{I}_\rho(\Delta)$ as the probability that a measurement of $\mathcal{I}$ results in an outcome in Δ when the system is in the state ρ. Notice that since $\overline{\mathcal{I}} = \mathcal{I}(\Omega_{\mathcal{I}})$ is a channel, we have that

$$\Phi_\rho^{\mathcal{I}}(\Omega_{\mathcal{I}}) = \operatorname{tr}\left[\overline{\mathcal{I}}(\rho)\right] = 1$$

so $\Phi_\rho^{\mathcal{I}}$ is a probability measure for every $\rho \in \mathcal{S}(H)$. If $\mathcal{J}$ is another instrument with outcome space $(\Omega_{\mathcal{J}}, \mathcal{F}_{\mathcal{J}})$, their *sequential product* $\mathcal{I} \circ \mathcal{J}$ of $\mathcal{I}$ then $\mathcal{J}$ is the instrument with outcome space $(\Omega_{\mathcal{I}} \times \Omega_{\mathcal{J}}, \mathcal{F}_{\mathcal{I}} \times \mathcal{F}_{\mathcal{J}})$ that satisfies

$$(\mathcal{I} \circ \mathcal{J})(\Delta \times \Gamma)(\rho) = \mathcal{J}(\Gamma)\left(\mathcal{I}(\Delta)(\rho)\right)$$

for all $\Delta \in \mathcal{F}_{\mathcal{I}}, \Gamma \in \mathcal{F}_{\mathcal{J}}, \rho \in \mathcal{S}(H)$. The *joint distribution* satisfies

$$\begin{aligned}\Phi_\rho^{\mathcal{I}\circ\mathcal{J}}(\Delta \times \Gamma) &= \operatorname{tr}\left[(\mathcal{I} \circ \mathcal{J})(\Delta \times \Gamma)(\rho)\right] \\ &= \operatorname{tr}\left[\mathcal{J}(\Gamma)\left(\mathcal{I}(\Delta)\right)(\rho)\right]\end{aligned}$$

We define $\mathcal{J}$ *conditioned by* $\mathcal{I}$ to be the instrument $(\mathcal{J} \mid \mathcal{I})$ with outcome space $(\Omega_{\mathcal{J}}, \mathcal{F}_{\mathcal{J}})$ given by

$$(\mathcal{J} \mid \mathcal{I})(\Gamma)(\rho) = \mathcal{J}(\Gamma)\left[\overline{\mathcal{I}}(\rho)\right] = \mathcal{J}(\Gamma)\left[\mathcal{I}(\Omega_{\mathcal{I}})(\rho)\right]$$

If $\mathcal{I} \in \mathcal{I}n(H)$ we have that $\mathcal{I}(\Delta) \in \mathcal{O}(H)$ and hence $\mathcal{I}(\Delta)^* \in \mathcal{O}^*(H)$ for all $\Delta \in \mathcal{F}_{\mathcal{I}}$. We call $\mathcal{I}^*(\Delta) = \mathcal{I}(\Delta)^*$ a *dual instrument*. Thus, $\mathcal{I}^*$ is a dual operation-valued measure on $(\Omega_{\mathcal{I}}, \mathcal{F}_{\mathcal{I}})$ satisfying $\mathcal{I}^*(\Delta)\colon \mathcal{E}(H) \to \mathcal{E}(H)$ for all $\Delta \in \mathcal{F}_{\mathcal{I}}$ and by Theorem 1

$$\mathcal{I}^*(\Omega_{\mathcal{I}})(I) = \overline{\mathcal{I}}(I) = I$$

Moreover, $\mathcal{I}^*$ is the unique dual instrument satisfying

$$\operatorname{tr}\left[\rho \mathcal{I}^*(\Delta)(a)\right] = \operatorname{tr}\left[\mathcal{I}(\Delta)(\rho)a\right] \tag{4}$$

for all $\rho \in \mathcal{S}(H), \Delta \in \mathcal{F}_{\mathcal{I}}, a \in \mathcal{E}(H)$. We denote the set of dual instruments by $\mathcal{I}n^*(H)$.

If $(\Omega_A, \mathcal{F}_A)$ is a measurable space, an *observable* A on H with *outcome space* $(\Omega_A, \mathcal{F}_A)$ is an effect-valued measure on $\mathcal{F}_A$ satisfying $A(\Omega_A) = I$ [1, 2, 5, 6, 11]. We denote the set of observables on H by $\mathcal{O}b(H)$. If $A \in \mathcal{O}b(H)$, we interpret $A(\Delta)$ as the effect resulting from A having an outcome in $\Delta \in \mathcal{F}_A$ when A is measured. The probability that A results in an outcome in Δ when the system is in the state $\rho \in \mathcal{S}(H)$ is given by $\Phi_\rho^A(\Delta) = \operatorname{tr}\left[\rho A(\Delta)\right]$ and Φ_ρ^A is the *distribution* of A in the state ρ. If $\mathcal{I} \in \mathcal{I}n(H)$, the unique observable $\widehat{\mathcal{I}} \in \mathcal{O}(H)$ *measured* by $\mathcal{I}$ has outcome space $(\Omega_{\mathcal{I}}, \mathcal{F}_{\mathcal{I}})$ and satisfies [1, 2, 5]

$$\Phi_\rho^{\widehat{\mathcal{I}}}(\Delta) = \operatorname{tr}\left[\rho\widehat{\mathcal{I}}(\Delta)\right] = \operatorname{tr}\left[\mathcal{I}(\Delta)(\rho)\right] = \Phi_\rho^{\mathcal{I}}(\Delta) \tag{5}$$

Applying (4) and (5) we obtain

$$\operatorname{tr}\left[\rho\widehat{\mathcal{I}}(\Delta)\right] = \operatorname{tr}\left[\rho\widehat{\mathcal{I}}(\Delta)I\right] = \operatorname{tr}\left[\mathcal{I}(\Delta)(\rho)I\right] = \operatorname{tr}\left[\rho\mathcal{I}^*(\Delta)(I)\right]$$

for all $\rho \in \mathcal{S}(H)$. Hence, for all $\Delta \in \mathcal{F}_{\mathcal{I}}$ we have

$$\widehat{\mathcal{I}}(\Delta) = \mathcal{I}^*(\Delta)(I) \tag{6}$$

As with operations, although $\mathcal{I} \in \mathcal{I}n(H)$ measures an unique observable $\widehat{\mathcal{I}}$, an observable is measured by many instruments. Moreover, we interpret an instrument $\mathcal{I}$ as an apparatus that can be employed to measure the observable $\widehat{\mathcal{I}}$. The next result follows from Theorem 2.

Theorem 7. (i) If $0 \le \lambda_i \le 1$ with $\sum \lambda_i = 1$ and $\mathcal{I}_i \in \mathcal{I}n(H)$, then $\sum \lambda_i \mathcal{I}_i \in \mathcal{I}n(H)$, $(\sum \lambda_i \mathcal{I}_i)^* = \sum \lambda_i \mathcal{I}_i^*$ and $(\sum \lambda_i \mathcal{I}_i)^\wedge = \sum \lambda_i \widehat{\mathcal{I}}_i$. (ii) If we also have $0 \le \mu_j \le 1$ with $\sum \mu_j = 1$ and $\mathcal{J}_j \in \mathcal{I}n(H)$, then

$$\left(\sum \lambda_i \mathcal{I}_i\right) \circ \left(\sum \mu_j \mathcal{J}_j\right) = \sum_{i,j} \lambda_i \mu_j \mathcal{I}_i \circ \mathcal{J}_j$$

and a similar result holds where $\mathcal{I}_i, \mathcal{J}_j \in \mathcal{I}n^*(H)$. (iii) If $\mathcal{I}, \mathcal{J} \in \mathcal{I}n(H)$, then $(\mathcal{I} \circ \mathcal{J})^* = \mathcal{J}^* \circ \mathcal{I}^*$ and $(\mathcal{I} \circ \mathcal{J})^\wedge = \mathcal{I}^*(\widehat{\mathcal{J}})$.

Let $A, B \in \mathcal{O}b(H)$ and let $\mathcal{I} \in \mathcal{I}n(H)$ satisfy $\widehat{\mathcal{I}} = A$. We define the *sequential product of A then B relative to $\mathcal{I}$* as the observable with outcome space $(\Omega_A \times \Omega_B, \mathcal{F}_A \times \mathcal{F}_B)$ given by $A\,[\mathcal{I}]\,B = \mathcal{I}^*(B)$. This is shorthand notation for

$$\begin{aligned}A\,[\mathcal{I}]\,B(\Delta \times \Gamma) &= \mathcal{I}^*(B)(\Delta \times \Gamma) = \mathcal{I}^*(\Delta)\left(B(\Gamma)\right) \\ &= \mathcal{I}(\Delta)^*\left(B(\Gamma)\right)\end{aligned} \tag{7}$$

Notice that $A\,[\mathcal{I}]\,B$ depends on the instrument $\mathcal{I}$ that measures A, but does not depend on the instrument measuring B. This is because B is measured second so its measurement cannot influence the A measurement. Applying (4) and (7), the distribution of $A\,[\mathcal{I}]\,B$ satisfies

$$\begin{aligned}\Phi_\rho^{A[\mathcal{I}]B}(\Delta \times \Gamma) &= \operatorname{tr}\left[\rho A\,[\mathcal{I}]\,B(\Delta \times \Gamma)\right] \\ &= \operatorname{tr}\left[\rho \mathcal{I}(\Delta)^*\left(B(\Gamma)\right)\right] \\ &= \operatorname{tr}\left[\mathcal{I}(\Delta)(\rho)B(\Gamma)\right]\end{aligned} \tag{8}$$

It follows from (8) that

$$\Phi_\rho^{A[\mathcal{I}]B}(\Delta \times \Omega_B) = \operatorname{tr}\left[\mathcal{I}(\Delta)(\rho)\right] = \Phi_\rho^A(\Delta)$$

for all $\rho \in \mathcal{S}(H)$, $\Delta \in \mathcal{F}_A$. We define *B conditioned by A relative to $\mathcal{I}$* as the observable with outcomes space $(\Omega_B, \mathcal{F}_B)$ given by

$$(B \mid \mathcal{I} \mid A)(\Gamma) = \overline{\mathcal{I}}^*\left(B(\Gamma)\right)$$

for all $\Gamma \in \mathcal{F}_B$. The distribution of $(B \mid \mathcal{I} \mid A)$ becomes

$$\begin{aligned}\Phi_\rho^{(B|\mathcal{I}|A)}(\Gamma) &= \operatorname{tr}\left[\rho\overline{\mathcal{I}}^*\left(B(\Gamma)\right)\right] = \operatorname{tr}\left[\overline{\mathcal{I}}(\rho)B(\Gamma)\right] \\ &= \operatorname{tr}\left[\mathcal{I}(\Omega_{\mathcal{I}})(\rho)B(\Gamma)\right] = \Phi_\rho^{A[\mathcal{I}]B}(\Omega_{\mathcal{I}} \times \Gamma)\end{aligned} \tag{9}$$

Notice that this idea has already been presented in the quantum formalism when we consider the updated state after the measurement of A results in an outcome in Δ. This updated state depends on the instrument $\mathcal{I}$ employed to measure A and is given by

$$\rho \mapsto \mathcal{I}(\Delta)\rho/\text{tr}\,[\mathcal{I}(\Delta)(\rho)] = [\mathcal{I}(\Delta)(\rho)]^{\sim}$$

Using a different instrument to measure A results in a different updated state in general. Even though $A\,[\mathcal{I}]\,B$ and $(B \mid \mathcal{I} \mid A)$ do not depend on the instrument $\mathcal{J}$ used to measure B, the next result gives an expression involving $\mathcal{J}$.

Lemma 8. Let $\mathcal{I}, \mathcal{J} \in \mathcal{I}n(H)$ satisfy $\widehat{\mathcal{I}} = A$, $\widehat{\mathcal{J}} = B$. (i) $A\,[\mathcal{I}]\,B = (\mathcal{I} \circ \mathcal{J})^{\wedge}$. (ii) $(B \mid \mathcal{I} \mid A) = (\mathcal{J} \mid \mathcal{I})^{\wedge}$.

Proof. (i) By Theorem 7(iii) we obtain

$$(\mathcal{I} \circ \mathcal{J})^{\wedge} = \mathcal{I}^*(\widehat{\mathcal{J}}) = \mathcal{I}^*(B) = A\,[\mathcal{I}]\,B$$

(ii) For all $\rho \in \mathcal{S}(H)$, $\Gamma \in \mathcal{F}_B$ we have

$$\begin{aligned}\text{tr}\left[\rho(\mathcal{J} \mid \mathcal{I})^{\wedge}(\Gamma)\right] &= \text{tr}\,[\rho(\mathcal{J} \mid \mathcal{I})^*(\Gamma)(I)] \\ &= \text{tr}\,[(\mathcal{J} \mid \mathcal{I})(\Gamma)(\rho)I] \\ &= \text{tr}\left\{\mathcal{J}(\Gamma)\left[\overline{\mathcal{I}}(\rho)\right]I\right\} \\ &= \text{tr}\left[\overline{\mathcal{I}}(\rho)\mathcal{J}^*(\Gamma)(I)\right] \\ &= \text{tr}\left[\overline{\mathcal{I}}(\rho)\widehat{\mathcal{J}}(\Gamma)\right] \\ &= \text{tr}\left[\overline{\mathcal{I}}(\rho)B(\Gamma)\right] \\ &= \text{tr}\left[\rho\overline{\mathcal{I}}^*(B(\Gamma))\right]\end{aligned}$$

Hence,

$$(\mathcal{J} \mid \mathcal{I})^{\wedge}(\Gamma) = \overline{\mathcal{I}}^*(B(\Gamma)) = (B \mid \mathcal{I} \mid A)(\Gamma)$$

for all $\Gamma \in \mathcal{F}_B$ so $(\mathcal{J} \mid \mathcal{I})^{\wedge} = (B \mid \mathcal{I} \mid A)$. □

If μ is a probability measure on $(\Omega, \mathcal{F})$ we call $\mathcal{I}_\mu(\Delta)(\rho) = \mu(\Delta)\rho$ for $\Delta \in \mathcal{F}$ an *identity instrument with measure* μ. Similarly, we define the *identity observable with measure* μ as $A_\mu(\Delta) = \mu(\Delta)I$ for $\Delta \in \mathcal{F}$. These are the simplest types of instruments and observables. The next theorem illustrates this theory in terms of these simple types. We first need an elementary lemma.

Lemma 9. If $A, B \in \mathcal{O}b(H)$, then $A\,[\mathcal{I}]\,B(\Omega_A \times \Gamma) = (B \mid \mathcal{I} \mid A)(\Gamma)$ and $A\,[\mathcal{I}]\,B(\Delta \times \Omega_B) = A(\Delta)$.

Proof. For all $\Gamma \in \mathcal{F}_B$ we obtain

$$\begin{aligned}A\,[\mathcal{I}]\,B(\Omega_A \times \Gamma) &= \mathcal{I}(\Omega_{\mathcal{I}})^*(B(\Gamma)) \\ &= \overline{\mathcal{I}}^*(B(\Gamma)) \\ &= (B \mid \mathcal{I} \mid A)(\Gamma)\end{aligned}$$

Moreover, for all $\Delta \in \mathcal{F}_A$ we obtain

$$\begin{aligned}A\,[\mathcal{I}]\,B(\Delta \times \Omega_B) &= \mathcal{I}(\Delta)^*(B(\Omega_B)) = \mathcal{I}(\Delta)^*I \\ &= \widehat{\mathcal{I}}(\Delta) = A(\Delta)\end{aligned}$$
□

Theorem 10. Let $\mathcal{I}_\mu$ be the identity instrument with measure μ. (i) $\mathcal{I}_\mu^*(\Delta)(a) = \mu(\Delta)(a)$ for all $\Delta \in \mathcal{F}$, $a \in \mathcal{E}(H)$ and $\widehat{\mathcal{I}}_\mu(\Delta) = \mu(\Delta)I$ is the identity observable with measure μ. (ii) If $A = \widehat{\mathcal{I}}_\mu$ and $B = \widehat{\mathcal{J}}$, then

$$A\left[\mathcal{I}_\mu\right]B(\Delta \times \Gamma) = B\,[\mathcal{J}]\,A(\Gamma \times \Delta) = \mu(\Delta)B(\Gamma)$$

(iii) If $A = \widehat{\mathcal{I}}_\mu$ and $B = \widehat{\mathcal{J}}$, then $\left(B \mid \mathcal{I}_\mu \mid A\right) = B$ and $(A \mid \mathcal{J} \mid B) = A$. (iv) If $A = \widehat{\mathcal{I}}_\mu$ and $B = \widehat{\mathcal{I}}_\nu$, then $A\left[\mathcal{I}_\mu\right]B$ is the identity observable with measure $\mu \times \nu$. (v) If $\mathcal{J} \in \mathcal{I}n(H)$, then $\left(\mathcal{J} \mid \mathcal{I}_\mu\right) = \mathcal{J}$, $\left(\mathcal{I}_\mu \mid \mathcal{J}\right)(\Delta)(\rho) = \mu(\Delta)\left[\overline{\mathcal{J}}(\rho)\right]$, $(\mathcal{J} \mid \mathcal{I}_\mu)^{\wedge} = \widehat{\mathcal{J}}$ and $(\mathcal{I}_\mu \mid \mathcal{J})^{\wedge} = \widehat{\mathcal{I}}_\mu$.

Proof. (i) For all $\rho \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$, $\Delta \in \mathcal{F}$ we have

$$\left[\rho\mathcal{I}_\mu^*(\Delta)(a)\right] = \text{tr}\left[\mathcal{I}_\mu(\Delta)(\rho)a\right] = \text{tr}\,[\mu(\Delta)\rho a] = \text{tr}\,[\rho\mu(\Delta)a]$$

Hence, $\mathcal{I}_\mu^*(\Delta)(a) = \mu(\Delta)a$. It follows that

$$\widehat{\mathcal{I}}_\mu(\Delta) = \mathcal{I}_\mu^*(\Delta)(I) = \mu(\Delta)I$$

(ii) Since $A\left[\mathcal{I}_\mu\right]B = \mathcal{I}_\mu^*(B)$, applying (i) we obtain

$$A\left[\mathcal{I}_\mu\right]B(\Delta \times \Gamma) = \mathcal{I}_\mu^*(\Delta)\,(B(\Gamma)) = \mu(\Delta)B(\Gamma)$$

Since $B\,[\mathcal{J}]\,A = \mathcal{J}^*(A)$ we obtain

$$\begin{aligned}B\,[\mathcal{J}]\,A(\Gamma \times \Delta) &= \mathcal{J}^*(\Gamma)\,(A(\Delta)) = \mathcal{J}^*(\Gamma)\,(\mu(\Delta)I) \\ &= \mu(\Delta)\mathcal{J}^*(\Gamma)(I) = \mu(\Delta)\widehat{\mathcal{J}}(\Gamma) = \mu(\Delta)B(\Gamma)\end{aligned}$$

(iii) Applying (ii) and Lemma 9 gives

$$\left(B \mid \mathcal{I}_\mu \mid A\right)(\Gamma) = A\left[\mathcal{I}_\mu\right]B(\Omega_A \times \Gamma) = \mu(\Omega_{\mathcal{I}})B(\Gamma) = B(\Gamma)$$

Hence, $\left(B \mid \mathcal{I}_\mu \mid A\right) = B$. Since $\widehat{\mathcal{J}} = B$, applying Lemma 9 gives

$$\begin{aligned}(A \mid \mathcal{J} \mid B)(\Delta) &= B\,[\mathcal{J}]\,A(\Omega_B \times \Delta) = \mu(\Delta)B(\Omega_B) \\ &= \mu(\Delta)I = A(\Delta)\end{aligned}$$

Hence, $(A \mid \mathcal{J} \mid B) = A$.
(iv) Applying (ii) gives

$$\begin{aligned}A\left[\mathcal{I}_\mu\right]B(\Delta \times \Gamma) &= \mu(\Delta)B(\Gamma) = \mu(\Delta)\widehat{\mathcal{I}}_\nu(\Gamma) \\ &= \mu(\Delta)\nu(\Gamma)I = (\mu \times \nu)(\Delta \times \Gamma)I\end{aligned}$$

and the result follows.
(v) For all $\Gamma \in \mathcal{F}_{\mathcal{J}}$, $\rho \in \mathcal{S}(H)$ we obtain

$$\left(\mathcal{J} \mid \mathcal{I}_\mu\right)(\Gamma)(\rho) = \mathcal{J}(\Gamma)\left[\overline{\mathcal{I}}_\mu(\rho)\right] = \mathcal{J}(\Gamma)(\rho)$$

Hence, $\left(\mathcal{J} \mid \mathcal{I}_\mu\right) = \mathcal{J}$. For all $\Delta \in \mathcal{F}$, $\rho \in \mathcal{S}(H)$ we obtain

$$\left(\mathcal{I}_\mu \mid \mathcal{J}\right)(\Delta)(\rho) = \mathcal{I}_\mu(\Delta)\left[\overline{\mathcal{J}}(\rho)\right] = \mu(\Delta)\left[\overline{\mathcal{J}}(\rho)\right]$$

Moreover, for all $\Gamma \in \mathcal{F}_{\mathcal{J}}$ we have

$$\left(\mathcal{J} \mid \mathcal{I}_\mu\right)^\wedge(\Gamma) = \overline{\mathcal{I}}_\mu^*\left(\widehat{\mathcal{J}}(\Gamma)\right) = \widehat{\mathcal{J}}(\Gamma)$$

Hence, $\left(\mathcal{J} \mid \mathcal{I}_\mu\right)^\wedge = \widehat{\mathcal{J}}$. Finally, we have for all $\Delta \in \mathcal{F}$ that

$$\begin{aligned}\left(\mathcal{I}_\mu \mid \mathcal{J}\right)^\wedge(\Delta) &= \overline{\mathcal{J}}^*\left(\widehat{\mathcal{I}}_\mu(\Delta)\right) = \overline{\mathcal{J}}^*\left[\mu(\Delta)I\right] \\ &= \mu(\Delta)\overline{\mathcal{J}}^*(I) = \mu(\Delta)I = \widehat{\mathcal{I}}_\mu(\Delta)\end{aligned}$$

so $\left(\mathcal{I}_\mu \mid \mathcal{J}\right)^\wedge = \widehat{\mathcal{I}}_\mu$. □

We can extend the definition of a Holevo operation to a Holevo instrument as follows. A *Holevo instrument with state α and observable A* has the form $\mathcal{H}_{(\alpha,A)}(\Delta)(\rho) = \mathrm{tr}\left[\rho A(\Delta)\right]\alpha$ for all $\Delta \in \Omega_A$.

Theorem 11. Let $\mathcal{H}_{(\alpha,A)}$ be a Holevo instrument. (i) $\mathcal{H}^*_{(\alpha,A)}(\Delta)(a) = \mathrm{tr}\,(\alpha a)A(\Delta)$ for all $\Delta \in \mathcal{F}_A$, $a \in \mathcal{E}(H)$ and $\widehat{\mathcal{H}}_{(\alpha,A)} = A$. (ii) $A\left[\mathcal{H}_{(\alpha,A)}\right]B(\Delta \times \Gamma) = \mathrm{tr}\left[\alpha B(\Gamma)\right]A(\Delta)$. (iii) $\left(B \mid \mathcal{H}_{(\alpha,A)} \mid A\right)(\Gamma) = \mathrm{tr}\left[\alpha B(\Gamma)\right]I$ which is an identity observable.

Proof. (i) For every $\rho \in \mathcal{S}(H)$, $\Delta \in \mathcal{F}_A$, $a \in \mathcal{E}(H)$, we obtain

$$\begin{aligned}\mathrm{tr}\left[\rho\mathcal{H}^*_{(\alpha,A)}(\Delta)(a)\right] &= \mathrm{tr}\left[\mathcal{H}_{(\alpha,A)}(\Delta)(\rho)a\right] \\ &= \mathrm{tr}\left\{\mathrm{tr}\left[\rho A(\Delta)\right]\alpha a\right\} \\ &= \mathrm{tr}\left[\rho A(\Delta)\right]\mathrm{tr}\,(\alpha a) \\ &= \mathrm{tr}\left\{\rho\,\mathrm{tr}\,(\alpha a)A(\Delta)\right\}\end{aligned}$$

Hence, $\mathcal{H}^*_{(\alpha,A)}(\Delta)(a) = \mathrm{tr}\,(\alpha a)A(\Delta)$. Moreover,

$$\widehat{\mathcal{H}}_{(\alpha,A)}(\Delta) = \mathcal{H}^*_{(\alpha,A)}(\Delta)I = A(\Delta)$$

for all $A \in \mathcal{F}_A$ so $\widehat{\mathcal{H}}_{(\alpha,A)} = A$.
(ii) Applying (i) we have

$$\begin{aligned}A\left[\mathcal{H}_{(\alpha,A)}\right]B(\Delta \times \Gamma) &= \mathcal{H}^*_{(\alpha,A)}(B)(\Delta \times \Gamma) \\ &= \mathcal{H}^*_{(\alpha,A)}(\Delta)\left(B(\Gamma)\right) \\ &= \mathrm{tr}\left[\alpha B(\Gamma)\right]A(\Delta)\end{aligned}$$

(iii) Applying Lemma 9 and (ii) give

$$\begin{aligned}\left(B \mid \mathcal{H}_{(\alpha,A)} \mid A\right)(\Gamma) &= A\left[\mathcal{H}_{(\alpha,A)}\right]B(\Omega_A \times \Gamma) \\ &= \mathrm{tr}\left[\alpha B(\Gamma)\right]A(\Omega_A) \\ &= \mathrm{tr}\left[\alpha B(\Gamma)\right]I\end{aligned}$$ □

An instrument $\mathcal{I}$ is *state constant* if $\mathcal{I}(\Delta)(\rho_1) = \mathcal{I}(\Delta)(\rho_2)$ for all $\rho_1, \rho_2 \in \mathcal{S}(H)$, $\Delta \in \mathcal{F}_{\mathcal{I}}$. If $\mathcal{J} \in \mathcal{I}n(H)$, $\alpha \in \mathcal{S}(H)$, we define the *$\alpha$-state constant* instrument $\mathcal{J}_\alpha$ by $\mathcal{J}_\alpha(\Delta)(\rho) = \mathcal{J}(\Delta)(\alpha)$ for all $\Delta \in \mathcal{F}_{\mathcal{J}}$, $\rho \in \mathcal{S}(H)$. It follows that $\mathcal{I}$ is state constant if and only if $\mathcal{I} = \mathcal{J}_\alpha$ for some $\mathcal{J} \in \mathcal{I}n(H)$, $\alpha \in \mathcal{S}(H)$. For example, the α-state constant instrument $\left[\mathcal{H}_{(\beta,A)}\right]_\alpha$ is given by

$$\left[\mathcal{H}_{(\beta,A)}\right]_\alpha(\Delta)(\rho) = \mathcal{H}_{(\beta,A)}(\Delta)(\alpha) = \mathrm{tr}\left[\alpha A(\Delta)\right]\beta$$

for all $\Delta \in \mathcal{F}_A$, $\rho \in \mathcal{S}(H)$. Notice that $\mathcal{J}_\alpha$ can be extended by linearity to all $\mathcal{T}(H)$.

Theorem 12. If $\mathcal{I}, \mathcal{J} \in \mathcal{I}n(H)$, $\alpha \in \mathcal{S}(H)$, the following statements hold. (i) $\mathcal{J}^*_\alpha(\Delta)(a) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)a\right]I$ and $\widehat{\mathcal{J}}_\alpha$ is the identity observable $\widehat{\mathcal{J}}_\alpha(\Delta) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)\right]I$. (ii) $(\mathcal{I} \mid \mathcal{J}_\alpha) = \mathcal{I}_{\overline{\mathcal{J}}(\alpha)}$, $(\mathcal{J}_\alpha \mid \mathcal{I}) = \mathcal{J}_\alpha$. (iii) $(\mathcal{I} \mid \mathcal{J}_\alpha)^\wedge$ is the identity observable,

$$(\mathcal{I} \mid \mathcal{J}_\alpha)^\wedge(\Delta) = \mathrm{tr}\left[\overline{\mathcal{J}}(\alpha)\widehat{\mathcal{I}}(\Delta)\right]I$$

and $(\mathcal{J}_\alpha \mid \mathcal{I})^\wedge = \widehat{\mathcal{J}}_\alpha$. (iv) If $A = \widehat{\mathcal{J}}_\alpha$, then $A\left[\mathcal{J}_\alpha\right]B$ is the identity observable with measure $\mu(\Delta \times \Gamma) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)B(\Gamma)\right]$ and $(B \mid \mathcal{J}_\alpha \mid A)$ is the identity observable with measure $\mathrm{tr}\left[\overline{\mathcal{J}}(\alpha)B(\Gamma)\right]$. (v) $\left(\mathcal{J} \mid \mathcal{H}_{(\alpha,A)}\right) = \mathcal{J}_\alpha$.

Proof. (i) For all $\rho \in \mathcal{S}(H)$, $\Delta \in \mathcal{F}_{\mathcal{J}}$, $a \in \mathcal{E}(H)$, we have that

$$\begin{aligned}\mathrm{tr}\left[\rho\mathcal{J}^*_\alpha(\Delta)(a)\right] &= \mathrm{tr}\left[\mathcal{J}_\alpha(\Delta)(\rho)(a)\right] = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)a\right] \\ &= \mathrm{tr}\left\{\rho\,\mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)a\right]I\right\}\end{aligned}$$

Hence, $\mathcal{J}^*_\alpha(\Delta) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)a\right]I$. Moreover,

$$\widehat{\mathcal{J}}_\alpha(\Delta) = \mathcal{J}^*_\alpha(\Delta)(I) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)\right]I$$

(ii) For all $\Delta \in \mathcal{F}_{\mathcal{I}}$, $\rho \in \mathcal{S}(H)$ we have

$$\begin{aligned}(\mathcal{I} \mid \mathcal{J}_\alpha)(\Delta)(\rho) &= \mathcal{I}(\Delta)\left[\overline{\mathcal{J}}_\alpha(\rho)\right] \\ &= \mathcal{I}(\Delta)\left[\overline{\mathcal{J}}(\alpha)\right] = \mathcal{I}_{\overline{\mathcal{J}}(\alpha)}(\Delta)(\rho)\end{aligned}$$

Hence, $(\mathcal{I} \mid \mathcal{J}_\alpha) = \mathcal{I}_{\overline{\mathcal{J}}(\alpha)}$. Moreover, for all $\Delta \in \mathcal{F}_{\mathcal{J}}$, $\rho \in \mathcal{S}(H)$ we obtain

$$(\mathcal{J}_\alpha \mid \mathcal{I})(\Delta)(\rho) = \mathcal{J}_\alpha(\Delta)\left[\overline{\mathcal{I}}(\rho)\right] = \mathcal{J}(\Delta)(\alpha) = \mathcal{J}_\alpha(\Delta)(\rho)$$

Thus, $(\mathcal{J}_\alpha \mid \mathcal{I}) = \mathcal{J}_\alpha$.
(iii) For all $\Delta \in \mathcal{F}_{\mathcal{I}}$ we obtain

$$(\mathcal{I} \mid \mathcal{J}_\alpha)^\wedge(\Delta) = \overline{\mathcal{J}}^*_\alpha\left[\widehat{\mathcal{I}}(\Delta)\right] = \mathrm{tr}\left[\overline{\mathcal{J}}(\alpha)\widehat{\mathcal{I}}(\Delta)\right]I$$

Applying (ii) gives $(\mathcal{J}_\alpha \mid \mathcal{I})^\wedge = \widehat{\mathcal{J}}_\alpha$.
(iv) For $\Delta \in \mathcal{F}_A$, $\Gamma \in \mathcal{F}_B$, applying (i) we obtain

$$A\left[\mathcal{J}_\alpha\right]B(\Delta \times \Gamma) = \mathcal{J}^*_\alpha(\Delta)\left(B(\Gamma)\right) = \mathrm{tr}\left[\mathcal{J}(\Delta)(\alpha)B(\Gamma)\right]I$$

Moreover, for all $\Gamma \in \mathcal{F}_B$ we obtain by Lemma 9 that

$$(B \mid \mathcal{J}_\alpha \mid A)(\Gamma) = A\left[\mathcal{J}_\alpha\right]B(\Omega_A \times \Gamma) = \mathrm{tr}\left[\overline{\mathcal{J}}(\alpha)B(\Gamma)\right]I$$

(v) For all $\Delta \in \mathcal{F}_{\mathcal{J}}$, $\rho \in \mathcal{S}(H)$ we have

$$\begin{aligned}\left(\mathcal{J} \mid \mathcal{H}_{(\alpha,A)}\right)(\Delta)(\rho) &= \mathcal{J}(\Delta)\left[\overline{\mathcal{H}}_{(\alpha,A)}(\rho)\right] \\ &= \mathcal{J}(\Delta)(\alpha) = \mathcal{J}_\alpha(\Delta)(\rho)\end{aligned}$$

and the result follows. □

An instrument $\mathcal{I}$ is *repeatable* if $\text{tr}\,[\mathcal{I}(\Delta)(\mathcal{I}(\Delta)\rho)] = \text{tr}\,[\mathcal{I}(\Delta)(\rho)]$ for all $\Delta \in \mathcal{F}_{\mathcal{I}}, \rho \in \mathcal{S}(H)$ [2].

Theorem 13. The following statements are equivalent.
(i) $\mathcal{I}$ is repeatable.
(ii) $\widehat{\mathcal{I}}(\Delta) = (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta \times \Delta)$ for all $\Delta \in \mathcal{F}_{\mathcal{I}}$.
(iii) $(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) = 0$ whenever $\Delta_1 \cap \Delta_2 = \emptyset$.
(iv) $\mathcal{I} \circ \mathcal{I}(\Delta_1 \times \Delta_2) = 0$ whenever $\Delta_1 \cap \Delta_2 = \emptyset$.
(v) $(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) = \widehat{\mathcal{I}}(\Delta_1 \cap \Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{F}_{\mathcal{I}}$.
(vi) $\widehat{\mathcal{I}}\,[\mathcal{I}]\,\widehat{\mathcal{I}}(\Delta_1 \times \Delta_2) = \widehat{\mathcal{I}}(\Delta_1 \cap \Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{F}$.

Proof. (i)⇔(ii) If $\mathcal{I}$ is repeatable, then for all $\Delta \in \mathcal{F}_{\mathcal{I}}$, $\rho \in \mathcal{S}(H)$ we obtain by Lemma 8 that

$$\begin{aligned}\text{tr}\left[\rho\widehat{\mathcal{I}}(\Delta)\right] &= \text{tr}\,[\mathcal{I}(\Delta)(\rho)] \\ &= \text{tr}\,[\mathcal{I}(\Delta)\,(\mathcal{I}(\Delta)\rho))] \\ &= \text{tr}\left[\mathcal{I}(\Delta)(\rho)\widehat{\mathcal{I}}(\Delta)\right] \\ &= \text{tr}\left[\rho\mathcal{I}^*(\Delta)\left(\widehat{\mathcal{I}}(\Delta)\right)\right] \\ &= \text{tr}\left[\rho(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta \times \Delta)\right]\end{aligned}$$

Hence, $\widehat{\mathcal{I}}(\Delta) = (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta \times \Delta)$ for all $\Delta \in \mathcal{F}_{\mathcal{I}}$. This also implies the converse.
(iii)⇔(ii) Suppose (ii) holds and $\Delta_1 \cap \Delta_2 = \emptyset$. Then

$$\begin{aligned}\widehat{\mathcal{I}}(\Delta_1) + \widehat{\mathcal{I}}(\Delta_2) = \widehat{\mathcal{I}}(\Delta_1 \cup \Delta_2) = (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \cup \Delta_2 \times \Delta_1 \cup \Delta_2) &= (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_1 \cup \Delta_2 \times \Delta_2 \cup \Delta_1 \times \Delta_2 \cup \Delta_2 \times \Delta_1) \\ &= (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_1) + (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_2 \times \Delta_2) + (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) + (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_2 \times \Delta_1) \\ &= \widehat{\mathcal{I}}(\Delta_1) + \widehat{\mathcal{I}}(\Delta_2) + (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) + (\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_2 \times \Delta_1)\end{aligned}$$

It follows that $(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) = 0$. To show the converse, suppose (iii) holds. Denoting the complement of Δ by Δ', we obtain

$$\text{tr}\,[\mathcal{I}(\Delta)(\rho)] = \text{tr}\,[\mathcal{I}(\Delta \cup \Delta')(\mathcal{I}(\Delta)\rho)] = \text{tr}\,[(\mathcal{I}(\Delta) + \mathcal{I}(\Delta'))\,(\mathcal{I}(\Delta)\rho)] = \text{tr}\,[\mathcal{I}(\Delta)\,(\mathcal{I}(\Delta)\rho)] + \text{tr}\,[\mathcal{I}(\Delta')\,(\mathcal{I}(\Delta)\rho)]$$

Applying Lemma 8 gives

$$\text{tr}\,[\mathcal{I}(\Delta')\,(\mathcal{I}(\Delta)\rho)] = \text{tr}\left[\mathcal{I}(\Delta)(\rho)\widehat{\mathcal{I}}(\Delta')\right] = \text{tr}\left[\rho\mathcal{I}^*(\Delta)\left(\widehat{\mathcal{I}}(\Delta')\right)\right] = \text{tr}\left[\rho(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta \times \Delta')\right] = 0$$

Hence, $\text{tr}\,[\mathcal{I}(\Delta)(\rho)] = \text{tr}\,[\mathcal{I}(\Delta)\,(\mathcal{I}(\Delta)\rho)]$ so (i) and (ii) hold.
(iii)⇔(iv) If $\mathcal{J}(\Gamma) = 0$ then

$$\text{tr}\left[\rho\widehat{\mathcal{J}}(\Gamma)\right] = \text{tr}\,[\mathcal{J}(\Gamma)(\rho)] = 0$$

so $\widehat{\mathcal{J}}(\Gamma) = 0$. Conversely, if $\widehat{\mathcal{J}}(\Gamma) = 0$, then $\text{tr}\,[\mathcal{J}(\Gamma)(\rho)] = 0$ for all $\rho \in \mathcal{S}(H)$. Since $\mathcal{J}(\Gamma)(\rho)$ is positive, it follows that $\mathcal{J}(\Gamma)(\rho) = 0$ for all ρ so $\mathcal{J}(\Gamma) = 0$. Replacing $\mathcal{J}$ with $\mathcal{I} \circ \mathcal{I}$ gives the result.
(iii)⇔(v) Suppose (iii) holds. Since

$$\Delta_1 \times \Delta_2 = \Delta_1 \times \left[(\Delta_2 \cap \Delta_1) \cup (\Delta_2 \cap \Delta_1')\right] = [\Delta_1 \times (\Delta_2 \cap \Delta_1)] \cup \left[\Delta_1 \times (\Delta_2 \cap \Delta_1')\right]$$

and

$$[\Delta_1 \times (\Delta_2 \cap \Delta_1)] \cup \left[\Delta_1 \times (\Delta_2 \cap \Delta_1')\right] = \Delta_1 \cap (\Delta_2 \cap \Delta_1') = \emptyset$$

we have by (iii) that

$$(\mathcal{I} \circ \mathcal{I})^{\wedge}\left[\Delta_1 \times (\Delta_2 \cap \Delta_1')\right] = 0$$

Since (iii)⇒(ii) we obtain

$$\begin{aligned}(\mathcal{I} \circ \mathcal{I})^{\wedge}(\Delta_1 \times \Delta_2) &= (\mathcal{I} \circ \mathcal{I})^{\wedge}\,[\Delta_1 \times (\Delta_2 \cap \Delta_1)] \\ &= (\mathcal{I} \circ \mathcal{I})^{\wedge}\left[(\Delta_1 \cap \Delta_2) \cup (\Delta_1 \cap \Delta_2') \times (\Delta_2 \cap \Delta_1)\right] \\ &= (\mathcal{I} \circ \mathcal{I})^{\wedge}\left[(\Delta_1 \cap \Delta_2) \times (\Delta_1 \cap \Delta_2) \cup (\Delta_1 \cap \Delta_2') \times (\Delta_1 \cap \Delta_2)\right] \\ &= (\mathcal{I} \circ \mathcal{I})^{\wedge}\,[(\Delta_1 \cap \Delta_2) \times (\Delta_1 \cap \Delta_2)] = \widehat{\mathcal{I}}(\Delta_1 \cap \Delta_2)\end{aligned}$$

Clearly, (v) implies (iii).
(v)⇔(vi) This follows because by Lemma 8 we have that $\widehat{\mathcal{I}}\,[\mathcal{I}]\,\widehat{\mathcal{I}} = (\mathcal{I}\circ\mathcal{I})^\wedge$. Reversing the implication shows that (vi) implies (i). Alternatively, since $\widehat{\mathcal{I}}\,[\mathcal{I}]\,\widehat{\mathcal{I}} = (\mathcal{I}\circ\mathcal{I})^\wedge$, letting $\Delta_1 = \Delta_2 = \Delta$ we obtain from (v) that

$$\widehat{\mathcal{I}}(\Delta) = \widehat{\mathcal{I}}\,[\mathcal{I}]\,\widehat{\mathcal{I}}(\Delta\times\Delta) = (\mathcal{I}\circ\mathcal{I})^\wedge(\Delta\times\Delta) \qquad \square$$

Corollary 14. The following statements are equivalent. (i) $\mathcal{I}$ is repeatable. (ii) $\mathcal{I}^*(\Delta)I = \mathcal{I}^*(\Delta)\,[\mathcal{I}^*(\Delta)I]$ for all $\Delta\in\mathcal{F}_\mathcal{I}$. (iii) $\mathcal{I}^*(\Delta_1)\,[\mathcal{I}^*(\Delta_2)I] = 0$ whenever $\Delta_1\cap\Delta_2 = \emptyset$. (iv) $\mathcal{I}^*(\Delta_1)\,[\mathcal{I}^*(\Delta_2)I] = \mathcal{I}^*(\Delta_1\cap\Delta_2)I$ for all $\Delta_1,\Delta_2\in\mathcal{F}_\mathcal{I}$.

Proof. By Theorem 13(ii), $\mathcal{I}$ is repeatable if and only if for all $\Delta\in\mathcal{F}_\mathcal{I}$ we have

$$\begin{aligned}\mathcal{I}^*(\Delta)I &= \widehat{\mathcal{I}}(\Delta) = (\mathcal{I}\circ\mathcal{I})^*(\Delta\times\Delta)\\ &= \mathcal{I}^*(\Delta)\left[\widehat{\mathcal{I}}(\Delta)\right] = \mathcal{I}^*\,[\mathcal{I}^*(\Delta)I]\end{aligned}$$

By Theorem 13(iii), $\mathcal{I}$ is repeatable if and only if whenever $\Delta_1\cap\Delta_2 = \emptyset$ we have

$$\mathcal{I}^*(\Delta_1)\,[\mathcal{I}^*(\Delta_2)I] = (\mathcal{I}\circ\mathcal{I})^\wedge(\Delta_1\times\Delta_2) = 0$$

By Theorem 13(v), $\mathcal{I}$ is repeatable if and only if for all $\Delta_1,\Delta_2\in\mathcal{F}_\mathcal{I}$ we have

$$\begin{aligned}\mathcal{I}^*(\Delta_1)\,[\mathcal{I}^*(\Delta_2)I] &= (\mathcal{I}\circ\mathcal{I})^\wedge(\Delta_1\times\Delta_2) = \widehat{\mathcal{I}}(\Delta_1\cap\Delta_2)\\ &= \mathcal{I}^*(\Delta_1\cap\Delta_2)I\end{aligned} \qquad \square$$

4 Finite Instruments and Observables

We now consider finite instruments and observables. One of the main advantages of the finite case is that we can introduce Lüders instruments [3,4] which do not seem to exist in the infinite case. Although finiteness is a strong assumption, it is general enough to include quantum computation and information theory [2,5,15]. For a finite set $\Omega = \{x_1, x_2, \ldots, x_n\}$ we assume that the corresponding σ-algebra is 2^Ω so the outcome space is specified by Ω and we need not mention the σ-algebra. A *finite instrument* with outcome space Ω corresponds to a set

$$\mathcal{I} = \{\mathcal{I}_{x_1}, \mathcal{I}_{x_2}, \ldots, \mathcal{I}_{x_n}\} \subseteq \mathcal{O}(H)$$

for which $\overline{\mathcal{I}} = \sum_{i=1}^n \mathcal{I}_{x_i}$ is a channel. We then define $\mathcal{I}(\Delta) = \sum_{x_i\in\Delta}\mathcal{I}_{x_i}$ for all $\Delta\subseteq\Omega$ so $\Delta\mapsto\mathcal{I}(\Delta)$ becomes an instrument [2, 5, 11, 12]. Similarly, a *finite observable* with outcome space Ω corresponds to a set $A = \{A_{x_1}, A_{x_2}, \ldots A_{x_n}\}\subseteq\mathcal{E}(H)$ for which $\sum_{i=1}^n A_{x_i} = I$. We again define $A(\Delta) = \sum_{x_i\in\Delta} A_{x_i}$ and $\Delta\mapsto A(\Delta)$ becomes an observable. As before, an instrument $\mathcal{I}$ measures a unique observable $\widehat{\mathcal{I}}$ that satisfies $\mathrm{tr}\,(\rho\widehat{\mathcal{I}}_{x_i}) = \mathrm{tr}\,[\mathcal{I}_{x_i}(\rho)]$, $i = 1,2,\ldots,n$, $\rho\in\mathcal{S}(H)$. Of course, this is equivalent to

$$\mathrm{tr}\left[\rho\widehat{\mathcal{I}}(\Delta)\right] = \mathrm{tr}\,[\mathcal{I}(\Delta)(\rho)]$$

for all $\Delta\subseteq\Omega$, $\rho\in\mathcal{S}(H)$. For conciseness, we use the notion

$$\mathcal{I}(x) = \mathcal{I}(\{x\}) = \mathcal{I}_x$$

Theorem 15. A finite instrument $\mathcal{I}$ is repeatable if and only if

$$\mathrm{tr}\,[\mathcal{I}_x(\rho)] = \mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))]$$

for all $x\in\Omega_\mathcal{I}$, $\rho\in\mathcal{S}(H)$.

Proof. If $\mathcal{I}$ is repeatable, then

$$\begin{aligned}\mathrm{tr}\,[\mathcal{I}_x(\rho)] &= \mathrm{tr}\,[\mathcal{I}(x)(\rho)] = \mathrm{tr}\,[\mathcal{I}(x)(\mathcal{I}(x)(\rho))]\\ &= \mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))]\end{aligned}$$

for all $x\in\Omega_\mathcal{I}$, $\rho\in\mathcal{S}(H)$. Conversely, suppose $\mathrm{tr}\,[\mathcal{I}_x(\rho)] = \mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))]$ holds. Since

$$\sum_y \mathrm{tr}\left[\mathcal{I}_y(\mathcal{I}_x(\rho))\right] = \mathrm{tr}\left[\overline{\mathcal{I}}(\mathcal{I}_x(\rho))\right] = \mathrm{tr}\,[\mathcal{I}_x(\rho)]$$

we conclude that $\sum_{y\neq x}\mathrm{tr}\left[\mathcal{I}_y(\mathcal{I}_x(\rho))\right] = 0$ so $\mathrm{tr}\left[\mathcal{I}_y(\mathcal{I}_x(\rho))\right] = 0$ for all $\rho\in\mathcal{S}(H)$ and $y\neq x$. We conclude that

$$\begin{aligned}\mathrm{tr}\,[\mathcal{I}(\Delta)(\mathcal{I}(\Delta)(\rho))] &= \mathrm{tr}\left[\left(\sum_{y\in\Delta}\mathcal{I}_y\right)\left(\sum_{x\in\Delta}\mathcal{I}_x(\rho)\right)\right]\\ &= \sum_{x,y\in\Delta}\mathrm{tr}\left[\mathcal{I}_y(\mathcal{I}_x(\rho))\right]\\ &= \sum_{x\in\Delta}\mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))]\\ &= \sum_{x\in\Delta}\mathrm{tr}\,[\mathcal{I}_x(\rho)]\\ &= \mathrm{tr}\,[\mathcal{I}(\Delta)(\rho)]\end{aligned}$$

for all $\Delta\subseteq\Omega_\mathcal{I}$, $\rho\in\mathcal{S}(H)$. Hence, $\mathcal{I}$ is repeatable. $\square$

The instrument $\mathcal{I}\circ\mathcal{J}$ and observables $A\,[\mathcal{I}]\,B$ are determined by their outcomes $(\mathcal{I}\circ\mathcal{J})_{(x,y)} = \mathcal{I}_x\circ\mathcal{J}_y$ and $(A\,[\mathcal{I}]\,B)_{(x,y)} = \mathcal{I}_x^*(B_y)$. The next result follows from Theorem 13.

Corollary 16. The following statements for a finite instrument $\mathcal{I}$ are equivalent. (i) $\mathcal{I}$ is repeatable. (ii) $\widehat{\mathcal{I}}_x = (\mathcal{I}\circ\mathcal{I})^\wedge_{(x,x)}$ for all $x\in\Omega_\mathcal{I}$. (iii) $(\mathcal{I}\circ\mathcal{I})^\wedge_{(x,y)} = 0$ if $x\neq y$. (iv) $(\mathcal{I}\circ\mathcal{I})^\wedge_{(x,y)} = \widehat{\mathcal{I}}(\{x\}\cap\{y\})$ for all $x,y\in\Omega_\mathcal{I}$. (v) For all $x,y\in\Omega_\mathcal{I}$ we have

$$\left(\widehat{\mathcal{I}}\,[\mathcal{I}]\,\widehat{\mathcal{I}}\right)_{(x,y)} = \widehat{\mathcal{I}}(\{x\}\cap\{y\})$$

We now consider a generalization of a Holevo instrument for the finite case. If $A = \{A_x : x \in \Omega\}$ is a finite observable and $\alpha_x \in \mathcal{S}(H)$, $x \in \Omega$, then the instrument

$$[\mathcal{H}_{(\alpha,A)}]_x(\rho) = \mathrm{tr}\,(\rho A_x)\alpha_x$$

is called a (*finite*) *Holevo instrument with states* α_x *and observable* A. The instrument $\mathcal{H}_{(\alpha,A)}$ is also called a *conditional state preparator* [2].

Lemma 17. A Holevo instrument $\mathcal{H}_{(\alpha,A)}$ is repeatable if and only if $\mathrm{tr}\,(\alpha_x A_x) = 1$ for all x with $A_x \neq 0$.

Proof. For all $\rho \in \mathcal{S}(H)$, $x \in \Omega$, writing $\mathcal{I} = \mathcal{H}_{(\alpha,A)}$ we obtain

$$\begin{aligned}\mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))] &= \mathrm{tr}\,[\mathcal{I}_x(\mathrm{tr}\,(\rho A_x)\alpha_x)]\\ &= \mathrm{tr}\,(\rho A_x)\mathrm{tr}\,[\mathcal{I}_x(\alpha_x)]\\ &= \mathrm{tr}\,(\rho A_x)\mathrm{tr}\,(\alpha_x A_x)\end{aligned}$$

Hence, $\mathcal{I}$ is repeatable if and only if

$$\mathrm{tr}\,[\mathcal{I}_x(\mathcal{I}_x(\rho))] = \mathrm{tr}\,[\mathcal{I}_x(\rho)] = \mathrm{tr}\,(\rho A_x)$$

which is equivalent to $\mathrm{tr}\,(\rho A_x)\mathrm{tr}\,(\alpha_x A_x) = \mathrm{tr}\,(\rho A_x)$ for all $\rho \in \mathcal{S}(H)$, $x \in \Omega$. Choosing ρ such that $\mathrm{tr}\,(\rho A_x) \neq 0$ we conclude that $\mathrm{tr}\,(\alpha_x A_x) = 1$ for all x satisfying $A_x \neq 0$. □

In Lemma 17 we can choose $\alpha_x = |\psi_x\rangle\langle\psi_x|$ where $|\psi_x\rangle$ is a unit eigenvector for A_x. We now generalize Theorem 11 for finite Holevo instruments.

Theorem 18. (i) $(\mathcal{H}^*_{(\alpha,A)})_x(a) = \mathrm{tr}\,(\alpha_x a)A_x$ and $\widehat{\mathcal{H}}_{(\alpha,A)} = A$ so $\mathcal{H}_{(\alpha,A)}$ measures A. (ii) If $\mathcal{I} \in \mathcal{I}n(H)$ is finite, then $\mathcal{I} \circ \mathcal{H}_{(\alpha,A)}$ is a Holevo instrument with states α_y and observable $B_{(x,y)} = \mathcal{I}^*_x(A_y)$. (iii) If $\mathcal{I} \in \mathcal{I}n(H)$ is finite, then $\mathcal{H}_{(\alpha,A)} \circ \mathcal{I}$ is a Holevo instrument with states $\mathcal{I}_y(\alpha_x)^{\sim}$ where $\mathrm{tr}\left[\mathcal{I}_y(\alpha_x)\right] \neq 0$ and observable $B_{(x,y)} = \mathrm{tr}\left[\mathcal{I}_y(\alpha_x)\right]A_x$. (iv) $\mathcal{H}_{(\beta,B)} \circ \mathcal{H}_{(\alpha,A)}$ is a finite Holevo instrument with states α_y and observable $C_{(x,y)} = \mathrm{tr}\,(\beta_x A_y)B_x$. (v) $\left(B \mid \mathcal{H}_{(\alpha,A)} \mid A\right)_y = \sum\limits_x \mathrm{tr}\,(\alpha_x B_y)A_x$.

Proof. (i) For every $\rho \in \mathcal{S}(H)$ $x \in \Omega_A$, $a \in \mathcal{E}(H)$ we have

$$\begin{aligned}\mathrm{tr}\left[\rho(\mathcal{H}^*_{(\alpha,A)})_x(a)\right] &= \mathrm{tr}\,[(\mathcal{H}_{(\alpha,A)})_x(\rho)a] = \mathrm{tr}\,[\mathrm{tr}\,(\rho A_x)\alpha_x a]\\ &= \mathrm{tr}\,(\rho A_x)\mathrm{tr}\,(\alpha_x a) = \mathrm{tr}\,[\rho\,\mathrm{tr}\,(\alpha_x a)A_x]\end{aligned}$$

Hence, $(\mathcal{H}^*_{(\alpha,A)})_x(a) = \mathrm{tr}\,(\alpha_x a)A_x$. Moreover,

$$(\widehat{\mathcal{H}}_{(\alpha,A)})_x = \mathrm{tr}\,(\mathcal{H}^*_{(\alpha,A)})_x(I) = A_x$$

so $\widehat{\mathcal{H}}_{(\alpha,A)} = A$.

(ii) For all $x \in \Omega_{\mathcal{I}}$, $y \in \Omega_A$, $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}(\mathcal{I} \circ \mathcal{H}_{(\alpha,A)})_{(x,y)}(\rho) &= \mathcal{I}_x \circ (\mathcal{H}_{(\alpha,A)})_y(\rho)\\ &= (\mathcal{H}_{(\alpha,A)})_y\,(\mathcal{I}_x(\rho))\\ &= \mathrm{tr}\left[\mathcal{I}_x(\rho)A_y\right]\alpha_y = \mathrm{tr}\left[\rho\left(\mathcal{I}^*_x(A_y)\right)\right]\alpha_y\end{aligned}$$

Notice that $B_{(x,y)} = \mathcal{I}^*_x(A_y)$ is an observable because $\mathcal{I}^*_x(A_y) \in \mathcal{E}(H)$ and

$$\begin{aligned}\sum_{x,y} B_{(x,y)} &= \sum_{x,y} \mathcal{I}^*_x(A_y)\\ &= \sum_x \mathcal{I}^*_x\left(\sum_y A_y\right)\\ &= \sum_x \mathcal{I}^*_x(I) = \mathcal{I}(\Omega)^*(I) = I\end{aligned}$$

(iii) For all $x \in \Omega_A$, $y \in \Omega_{\mathcal{I}}$, $\rho \in \mathcal{S}(H)$ we obtain

$$\begin{aligned}(\mathcal{H}_{(\alpha,A)} \circ \mathcal{I})_{(x,y)}(\rho) &= (\mathcal{H}_{(\alpha,A)})_x \circ \mathcal{I}_y(\rho)\\ &= \mathcal{I}_y\left[(\mathcal{H}_{(\alpha,A)})_x(\rho)\right]\\ &= \mathrm{tr}\,(\rho A_x)\mathcal{I}_y(\alpha_x)\\ &= \mathrm{tr}\left[\rho\,\mathrm{tr}\left(\mathcal{I}_y(\alpha_x)\right)A_x\right]\mathcal{I}_y(\alpha_x)^{\sim}\end{aligned}$$

Notice that $B_{(x,y)} = \mathrm{tr}\left[\mathcal{I}_y(\alpha_x)\right]A_x$ is an observable because $\mathrm{tr}\left[\mathcal{I}_y(\alpha_x)\right]A_x \in \mathcal{E}(H)$ and

$$\begin{aligned}\sum_{x,y} B_{(x,y)} &= \sum_{x,y} \mathrm{tr}\left[\mathcal{I}_y(\alpha_x)\right]A_x\\ &= \sum_x \mathrm{tr}\left[\sum_y \mathcal{I}_y(x)\right]A_x\\ &= \sum_x \mathrm{tr}\left[\overline{\mathcal{I}}(\alpha_x)\right]A_x\\ &= \sum_x A_x = I\end{aligned}$$

(iv) Applying (ii) we obtain for all $x \in \Omega_B$, $y \in \Omega_A$, $\rho \in \mathcal{S}(H)$ that

$$\begin{aligned}(\mathcal{H}_{(\beta,B)} \circ \mathcal{H}_{(\alpha,A)})_{(x,y)}(\rho) &= \mathrm{tr}\left[\rho(\mathcal{H}^*_{(\beta,B)})_x A_y\right]\alpha_y\\ &= \mathrm{tr}\left[\rho\,\mathrm{tr}\,(\beta_x A_y)B_x\right]\alpha_y\\ &= \mathrm{tr}\,(\beta_x A_y)\mathrm{tr}\,(\rho B_x)\alpha_y\\ &= \mathrm{tr}\left[\rho\,\mathrm{tr}\,(\beta_x A_y)B_x\right]\alpha_y\end{aligned}$$

Notice that $C_{(x,y)} = \mathrm{tr}\,(\beta_x A_y)B_x$ is an observable because $\mathrm{tr}\,(\beta_x A_y)B_x \in \mathcal{E}(H)$ and

$$\begin{aligned}\sum_{x,y} C_{(x,y)} &= \sum_{x,y} \mathrm{tr}\,(\beta_x A_y)B_x = \sum_x \mathrm{tr}\left(\beta_x \sum_y A_y\right)B_x\\ &= \sum_x \mathrm{tr}\,(\beta_x I)B_x = \sum_x B_x = I\end{aligned}$$

(v) For all $y \in \Omega_B$, applying (i) we obtain

$$(B \mid \mathcal{H}_{(\alpha,A)} \mid A)_y = \overline{\mathcal{H}}^*_{(\alpha,A)}(B_y) = \sum_x \operatorname{tr}(\alpha_x B_y) A_x \qquad \square$$

If $A = \{A_x : x \in \Omega_A\}$ is a finite observable, we define the corresponding *Lüders instrument* [3, 4, 9] with outcome space Ω_A by

$$\mathcal{L}^A_x(\rho) = A_x^{\frac{1}{2}} \rho A_x^{\frac{1}{2}}$$

for all $x \in \Omega_A$. We then have for all $\Delta \subseteq \Omega_A$ that

$$\mathcal{L}^A(\Delta) = \sum \left\{ A_x^{\frac{1}{2}} \rho A_x^{\frac{1}{2}} : x \in \Delta \right\}$$

We now generalize Theorem 5 to instruments.

Theorem 19. Let $A, B \in Ob(H)$, $\mathcal{J} \in \mathcal{I}n(H)$ be finite.
(i) $(\mathcal{L}^A)^*_x(a) = A_x^{\frac{1}{2}} a A_x^{\frac{1}{2}}$ and $(\mathcal{L}^A)^\wedge = A$ so $\mathcal{L}^A$ measures A.
(ii) $(\mathcal{L}^A \circ \mathcal{J})^\wedge_{(x,y)} = A_x^{\frac{1}{2}} \widehat{\mathcal{J}}_y A_x^{\frac{1}{2}}$ for all $x \in \Omega_A$, $y \in \Omega_{\mathcal{J}}$.
(iii) $(\mathcal{J} \circ \mathcal{L}^A)^\wedge_{(y,x)} = \mathcal{J}^*_y(A_x)$ for all $x \in \Omega_X$, $y \in \Omega_{\mathcal{J}}$.
(iv) $(\mathcal{J} \mid \mathcal{L}^A)^\wedge_y = \sum_{x \in \Omega_A} A_x^{\frac{1}{2}} \widehat{\mathcal{J}}_y A_x^{\frac{1}{2}}$.
(v) $(\mathcal{L}^A \mid \mathcal{J})^\wedge_x = \overline{\mathcal{J}}^*(A_x)$.
(vi) $\left(A \left[\mathcal{L}^A\right] B\right)_{(x,y)} = A_x^{\frac{1}{2}} B_y A_x^{\frac{1}{2}}$.
(vii) $\left(B \mid \mathcal{L}^A \mid A\right)_y = \sum_{x \in \Omega_A} A_x^{\frac{1}{2}} B_y A_x^{\frac{1}{2}}$.

Proof. (i) For all $x \in \Omega_A$, $a \in \mathcal{E}(H)$, $\rho \in \mathcal{S}(H)$ we have

$$\begin{aligned} \operatorname{tr}\left[\rho (\mathcal{L}^A)^*_x(a)\right] &= \operatorname{tr}\left[\mathcal{L}^A_x(\rho) a\right] \\ &= \operatorname{tr}(A_x^{\frac{1}{2}} \rho A_x^{\frac{1}{2}} a) \\ &= \operatorname{tr}\left[\rho A_x^{\frac{1}{2}} a A_x^{\frac{1}{2}}\right] \end{aligned}$$

Hence, $(\mathcal{L}^A)^*_x(a) = A_x^{\frac{1}{2}} a A_x^{\frac{1}{2}}$. Moreover,

$$(\mathcal{L}^A)^\wedge_x = (\mathcal{L}^A_x)^*(I) = A_x^{\frac{1}{2}} I A_x^{\frac{1}{2}} = A_x$$

Therefore, $(\mathcal{L}^A)^\wedge = A$ so $\mathcal{L}^A$ measures A.
(ii) By Theorem 5(iii) we obtain

$$(\mathcal{L}^A \circ \mathcal{J})^\wedge_{(x,y)} = (\mathcal{L}^A)^*_x(\widehat{\mathcal{J}}_y) = A^{\frac{1}{2}} \widehat{\mathcal{J}}_y A_x^{\frac{1}{2}}$$

(iii) Applying Theorem 5 we have the following

$$(\mathcal{J} \circ \mathcal{L}^A)^\wedge_{(y,x)} = \mathcal{J}^*_y(\widehat{\mathcal{L}^A_x}) = \mathcal{J}^*_y(A_x)$$

(iv) $(\mathcal{J} \mid \mathcal{L}^A)^\wedge_y = (\overline{\mathcal{L}}^A)^*(\widehat{\mathcal{J}}_y) = \sum_{x \in \Omega_A} A_x^{\frac{1}{2}} \widehat{\mathcal{J}}_y A_x^{\frac{1}{2}}$
(v) $(\mathcal{L}^A \mid \mathcal{J})^\wedge_x = \widehat{\mathcal{J}}^*(\mathcal{L}^A)^\wedge_x = \sum_{y \in \Omega_{\mathcal{J}}} \mathcal{J}^*_y(A_x)$.
(vi) $\left(A \left[\mathcal{L}^A\right] B\right)_{(x,y)} = (\mathcal{L}^A_x)^*(B_y) = A_x^{\frac{1}{2}} B_y A_x^{\frac{1}{2}}$.
(vii) $(B \mid \mathcal{L}^A \mid A)_y = \left(A \left[\mathcal{L}^A\right] B\right)(\Omega_A \times \{y\}) = \sum_{x \in \Omega_A} A_x^{\frac{1}{2}} B_y A_x^{\frac{1}{2}}$. $\square$

Corollary 20. Let $A, B \in \mathcal{O}(H)$, $\mathcal{J} \in \mathcal{I}n(H)$ be finite and let $\Delta \subseteq \Omega_A$, $\Gamma \subseteq \Omega_B$. Then the following statements hold.
(i) $(\mathcal{L}^A)^*(\Delta)(a) = \sum_{x \in \Delta} (A_x \square a)$.
(ii) $(\mathcal{L}^A \circ \mathcal{J})^\wedge(\Delta \times \Gamma) = \sum_{x \in \Delta} \left[A_x \square \widehat{\mathcal{J}}(\Gamma)\right]$.
(iii) $(\mathcal{J} \circ \mathcal{L}^A)^\wedge(\Gamma \times \Delta) = \mathcal{J}^*(\Gamma)\left[A(\Delta)\right]$.
(iv) $(\mathcal{J} \mid \mathcal{L}^A)^\wedge(\Gamma) = \sum_{x \in \Omega_A} (A_x \square \widehat{\mathcal{J}}(\Gamma))$
(v) $(\mathcal{L}^A \mid \mathcal{J})^\wedge(\Delta) = \overline{\mathcal{J}}^*\left[A(\Delta)\right]$.
(vi) $\left(A \left[\mathcal{L}^A\right] B\right)(\Delta \times \Gamma) = \sum_{x \in \Delta} \left[A_x \square B(\Gamma)\right]$.
(vii) $(B \mid \mathcal{L}^A \mid A)(\Gamma) = \sum_{x \in \Omega_A} \left[A_x \square B(\Gamma)\right]$.

We close by stating that a Lüders instrument is repeatable if and only if it is sharp [2].

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